

# ATOMIC SATURATION OF REDUCED POWERS

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ABSTRACT. Our aim was to generalize some theorems about the saturation of ultra-powers to reduced powers. Naturally, we deal via saturation for types consisting of atomic formulas. We succeed to generalize “the theory of dense linear is maximal and so is any pair  $(T, \Delta)$  which is  $\text{SOP}_3$  ( $\Delta$  consists of atomic or conjunction of atomic formulas). However,  $\text{SOP}_2$  is not enough, so the  $\mathfrak{p} = \mathfrak{t}$  theorem cannot be generalized in this case. Similarly the unique dual of cofinality.

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## Anotated Content

§0 Introduction, (labels x,z), pg.3

§1 Axiomatizing [Sh:c, Ch.VI,2.6], (label h), pg. 8

[We phrase and prove a theorem which axiomatize [Sh:c, Ch.VI,2.6]. The theorem there says that if  $D$  is a regular ultrafilter on  $I$  and for every model  $M$  of the theory of dense linear orders, the model  $M^I/D$  is  $\lambda^+$ -saturated, then  $D$  is  $\lambda^+$ -good and  $\lambda$ -regular.]

§2 Applying the axiomatized frame, (label c), pg.12

[The axiomatization in §1 can be phrased as a set of sentences, surprisingly moreover Horn ones (first order if  $\theta_{\mathbf{r}} = \aleph_0$ ). Now in this case we can straightforwardly derive [Sh:a, Ch.VI,2.6]. But because the axiomatization being Horn, we can now deal also with the  $(\lambda^+, \text{atomic})$ -saturation of reduced power. We then deal with infinitary logics and comments on models of Bounded Peano Arithmetic.]

§3 Criterion for atomic saturation of reduced powers, (label g), pg.18

[For a complete first order  $T$  we characterize when a filter  $D$  on  $I$  is such that  $M^I/D$  is  $(\lambda, \text{atomically})$ -saturated for every model  $M$  of  $T$ .]

§4 Counterexample, (label b), pg.23

[We prove that for reduced powers, the parallel of  $\mathfrak{t} \leq \mathfrak{p}$  in general fails, similarly the uniqueness of the dual cofinality.]

§5 The orders  $\leq_{\lambda}^{\text{fp}}$ , (label g,r), pg.32

[We define and prove the results on morality from the ultra-filter case to the filter case. By this we prove that some  $\text{SOP}_2$  pair are not  $\leq^{\text{fp}}$ -maximal (unlike the ultra-filter cases, see 5.10).

## § 0. INTRODUCTION

### § 0(A). Background, Questions and Answers.

We know much on saturation of ultrapowers, but considerably less on reduced powers. For transparency, let  $T$  denote a first order complete countable theory with elimination of quantifiers and  $M$  will denote a model of  $T$ . For  $D$  a regular filter on  $\lambda > \aleph_0$  we may ask: when  $M^\lambda/D$  is saturated? For  $D$  an ultrafilter, Keisler [Kei65] proves that this holds for every  $T$  iff  $D$  is  $\lambda^+$ -good iff this holds for  $T =$  theory of Boolean algebras, such  $T$  is called  $\leq_\lambda$ -maximal.

By [Sh:a, Ch.VI,2.6] the maximality holds for  $T =$  theory of dense linear orders or just any  $T$  with the strict order property and by [Sh:500], any  $T$  with the  $\text{SOP}_3$  is  $\leq_\lambda$ -maximal.

What about reduced powers for  $\lambda$ -regular filter  $D$  on  $\lambda$ ? By [Sh:17],  $M^\lambda/D$  is  $\lambda^+$ -saturated for every  $T$  iff  $D$  is  $\lambda^+$ -good and  $\mathcal{P}(\lambda)/D$  is a  $\lambda^+$ -saturated Boolean Algebra. Parallel results hold when we replace  $\lambda^+$ -saturated by  $(\lambda^+, \Sigma_{1+n}(\mathbb{L}_{\mathcal{T}(T)}))$ -saturated. We shall concentrate on  $(\lambda^+, \text{atomically})$ -saturated and the related partial order  $\leq_\lambda^{\text{rp}}$ , see definitions below.

Concerning ultra-powers, lately Malliaris-Shelah [MiSh:998] proves that  $T$  being  $\text{SOP}_2$  suffice for  $\leq_\lambda$ -maximality (and that  $\mathfrak{t} = \mathfrak{p}$ ) and, in later work [MiSh:1051], that at least for a relative  $\aleph_\lambda^*$  (see [Sh:500]) this is “iff”. Part of the proof is axiomatized by Malliaris-Shelah [MiSh:1070].

Note also that [Sh:1019] deals with saturation but only for ultra-powers by  $\theta$ -complete ultrafilters for  $\theta$  a compact cardinal; and also with  $\omega$ -ultra-limits.

Now what do we accomplish here?

First, in §1 we axiomatize the proof of [Sh:a, Ch.VI,2.6], i.e. we define when  $\mathbf{r} = (M, \Delta)$  is an RSP and for it prove that the relevant model  $N_{\mathbf{r}}$  is  $(\min\{\mathfrak{p}_{\mathbf{r}}, \mathfrak{t}_{\mathbf{r}}\}, \Delta)$ -saturated. Second, in §2 we prove, of course, that [Sh:a, Ch.VI,2.6] follows, but also we show that the axiomatization of RSP is by Horn sentences. Hence we can apply it to reduced powers. So  $T$  is  $\leq_\lambda^{\text{rp}}$ -maximal if  $T = \text{Th}(\mathbb{Q}, <)$  and for  $T$  having the  $\text{SOP}_3$ ; lastly we comment on models of Peano Arithmetic.

In §3 we try to sort out when for models of  $T$  we get the relevant atomic saturation.

Can we generalize also results [MiSh:998] to reduced powers? The main result of §4 says that no. We also sort out the parallel of goodness, excellency and morality for filters and atomic saturation for reduced powers. In hopeful continuation, we consider parallel statements for infinite logics (see [Sh:1019]); also we consider non-maximality.

Note that by 2.10

**Conclusion 0.1.** *If  $(T, \Delta)$  has the  $\text{SOP}_3$ , then it is  $\leq_\lambda^{\text{rp}}$ -maximal.*

*Question 0.2.* Do we have: if  $D$  is  $(\lambda_2, T)$ -good and regular then  $D$  is  $(\lambda_1, T)$ -good when  $\lambda_1 < \lambda_2$  (or more).

### § 0(B). Further Questions.

**Convention 0.3.** 1) Let  $T$  be a theory with elimination of quantifiers if not said otherwise. Let  $\text{Mod}_T$  be the class of models of  $T$ .

2) The main case is for  $T$  is a countable complete first order theory with elimination of quantifiers, moreover, with every formula equivalent to an atomic one.

So it is natural to ask

**Conjecture 0.4.**  $(T, \Delta)$  is  $\leq_{\text{rp}}$ -maximal iff  $(T, \Delta)$  has the  $\text{SOP}_3$ .

So which  $T$  (with elimination of quantifiers) are maximal under  $\triangleleft_{\lambda}^{\text{rp}}$ ? That is, when for every regular filter  $D$  on  $\lambda$ ,  $M^{\lambda}/D$  is  $(\lambda^+, \text{atomically})$ -saturated iff  $D$  is  $\lambda^+$ -good? Is  $T_{\text{eq}}$  maximal? As we have not proved this even for ultrafilters, the reasonable hope is that it will be easier to show non-maximality for  $\triangleleft_{\lambda}^{\text{rp}}$ . Also in light of [MiSh:1030] for simple theories we like to prove non-maximality with no large cardinals. We may hope to use just  $\text{NSOP}_2$ , but still it would not settle the problem of characterizing the maximal ones as, e.g.  $\text{SOP}_2 \equiv \text{SOP}_3$  is open; for such  $T$ ; for a pair  $(T, \varphi(\bar{x}, \bar{y}))$  they are different.

Note that for first order  $T$ , it makes sense to use  $\mu^+$ -saturated models and  $D$  is  $\mu^+$ -complete.

Also the “ $T$  stable” case should be resolved.

**Conjecture 0.5.**  $M^{\lambda}/D$  is  $(\aleph_0^{\lambda}/D, \text{atomically})$ -saturated when:

- (a)  $T$  a theory as in 0.3
- (b)  $T$  is stable without the fcp
- (c)  $D$  is a  $(\lambda, \aleph_0)$ -regular filter on  $\lambda$ .

*Remark 0.6.* Maybe given a  $1 - \varphi$ -type  $p \subseteq \{\varphi(x, \bar{a}) : \bar{a} \in {}^m(M^I/D)\}$  of cardinality  $\leq \lambda$  in  $M^I/D$ , we try just to find a dense set of  $A \in D^+$  such that in  $M^I/(D + A)$  the  $1 - \varphi$ -type is realized. Then continue; opaque.

### § 0(C). Preliminaries.

*Notation 0.7.* 1) Let  $\mathfrak{B}$  denote a Boolean algebra,  $\text{comp}(\mathfrak{B})$  its completion,  $\mathfrak{B}^+ = \mathfrak{B} \setminus \{0_{\mathfrak{B}}\}$ ,  $\text{uf}(\mathfrak{B})$  the set of ultrafilters on  $\mathfrak{B}$ ,  $\text{fil}(\mathfrak{B})$  the set of filters on  $\mathfrak{B}$ . For  $\mathbf{a} \in \mathbf{B}$  let  $\mathbf{a}^{\text{if}(\text{true})} = \mathbf{a}^{\text{if}(1)}$  be  $\mathbf{a}$  and let  $\mathbf{a}^{\text{if}(\text{false})} = \mathbf{a}^{\text{if}(0)}$  be  $1_{\mathfrak{B}} - \mathbf{a}$ .

2) For a model  $M$  let  $\tau_M = \tau(M)$  be its vocabulary.

Now about cuts (they are different than gaps, see [MiSh:1069]).

**Definition 0.8.** 1) For a partial order  $\mathcal{T} = (\mathcal{T}, \leq_{\mathcal{T}})$ , we say  $(C_1, C_2)$  is pre-cut when:

- (a)  $C_1 \cup C_2$  is a subset of  $\mathcal{T}$  linearly ordered by  $\leq_{\mathcal{T}}$
- (b) if  $a_1 \in C_1, a_2 \in C_2$  then  $a_1 \leq_{\mathcal{T}} a_2$
- (c) for no  $c \in \mathcal{T}$  do we have  $a_1 \in C_1 \rightarrow a_1 \leq_{\mathcal{T}} c$  and  $a_2 \in C_2 \Rightarrow c \leq_{\mathcal{T}} a_2$ .

2) Above we say  $(C_1, C_2)$  is a  $(\kappa_1, \kappa_2)$ -pre-cut when in addition:

- (d)  $C_1$  has cofinality  $\kappa_1$
- (e)  $C_2^*$ , the inverse of  $C_2$ , has cofinality  $\kappa_2$
- (f) so  $\kappa_1, \kappa_2$  are regular or 0 or 1.

- 3) We may replace  $C_\ell$  by a sequence  $\bar{a}_\ell$ , if not said otherwise such that  $\bar{a}_1$  is  $\leq_{\mathcal{T}}$ -increasing and  $\bar{a}_2$  is  $\leq_{\mathcal{T}}$ -decreasing.
- 4) We say  $(C_1, C_2)$  is a  $(\kappa_1, \kappa_2)$ -linear-cut of  $\mathcal{T}$  when it is a  $(\kappa_1, \kappa_2)$ -pre-cut and  $C_1 \cup C_2$  is downward closed, so natural for  $\mathcal{T}$  a tree.
- 5) We say  $(C_1, C_2)$  is a weak cut when (b),(c) of part (1) holds.
- 6) We may write cut instead of pre-cut.

*Remark 0.9.* If  $\mathcal{T}$  is a (model theoretic) tree,  $\kappa_2 > 0$  and  $(C_1, C_2)$  is a  $(\kappa_1, \kappa_2)$ -pre-cut then it induces one and only one  $(\kappa_1, \kappa_2)$ -linear-cut  $(C'_1, C'_2)$ , i.e. one satisfying  $C_1 \subseteq C'_1, C_2 \subseteq C'_2$  such that  $C_1 \cup C_2$  is cofinal in  $C'_1 \cup C'_2$ .

**Definition 0.10.** 1) We say  $M$  is fully  $(\lambda, \theta, \sigma, L)$ -saturated (may omit the fully); where  $L \subseteq \mathcal{L}(\tau_M)$  and  $\mathcal{L}$  is a logic; we may write  $\mathcal{L}$  if  $L = \mathcal{L}(\tau_M)$ , when:

- if  $\Gamma$  is a set of  $< \lambda$  formulas from  $L$  with parameters from  $M$  with  $< 1 + \sigma$  free variables, and  $\Gamma$  is  $(< \theta)$ -satisfiable in  $M$ , then  $\Gamma$  is realized in  $M$ .

- 2) We say “locally” when using one  $\varphi = \varphi(\bar{x}, \bar{y}) \in \mathcal{L}$  with  $\ell g(\bar{x}) < 1 + \sigma$ , i.e. all members of  $\Gamma$  have the form<sup>1</sup>  $\varphi(\bar{x}, \bar{b})$ .
- 3) Saying “locally/fully  $(\lambda, \mathcal{L})$ -saturated” the default values (i.e. we may omit) of  $\sigma$  is  $\sigma = \theta$ , of  $(\sigma, \theta)$  is  $\theta = \aleph_0 \wedge \sigma = \aleph_0$  and of  $\mathcal{L}$  is  $\mathbb{L}$  (first order logic) and of  $L$  is  $\mathcal{L}$ . Omitting  $\mathcal{L}$  means  $\mathbb{L}_{\theta, \theta}$ , omitting  $\lambda$  means  $\lambda = \|M\|$ .
- 4) If  $\varphi(\bar{x}, y) \in \mathcal{L}(\tau_M)$  and  $\bar{a} \in {}^{\ell g(\bar{y})}M$  then  $\varphi(M, \bar{a}) := \{\bar{b} \in {}^{\ell g(\bar{x})}M : M \models \varphi[\bar{b}, \bar{a}]\}$ .
- 5) Let  $\bar{x}_{[u]} = \langle x_s : s \in u \rangle$ .

**Definition 0.11.** Assume we are given a Boolean Algebra  $\mathfrak{B}$  usually complete and a model or a set  $M$  and  $D$  a filter on  $\text{comp}(\mathfrak{B})$ , the completion of  $\mathfrak{B}$ .

1) Let  $M^{\mathfrak{B}}$  be the set of partial functions  $f$  from  $\mathfrak{B}^+ := \mathfrak{B} \setminus \{0_{\mathfrak{B}}\}$  into  $M$  such that for some maximal antichain  $\langle a_i : i < i(*) \rangle$  of  $\mathfrak{B}$ ,  $\text{Dom}(f)$  includes  $\{a_i : i < i(*)\}$  and is included in<sup>2</sup>  $\{a \in \mathfrak{B}^+ : (\exists i)(a \leq a_i)\}$  and  $f$  is a function into  $M$  and  $f \upharpoonright \{a \in \text{Dom}(f) : a \leq a_i\}$  is constant for each  $i$ .

1A) Naturally for  $f_1, f_2 \in M^{\mathfrak{B}}$  we say  $f_1, f_2$  are  $D$ -equivalent, or  $f_1 = f_2 \pmod{D}$  when for some  $b \in D$  we have  $a_1 \in \text{Dom}(f_1) \wedge a_2 \in \text{Dom}(f_2) \wedge a_1 \cap a_2 \cap b > 0_{\mathfrak{B}} \Rightarrow f_1(a_1) = f_2(a_2)$ .

2) We define  $M^{\mathfrak{B}}/D$  naturally, as well as  $\text{TV}(\varphi(f_0, \dots, f_{n-1})) \in \text{comp}(\mathfrak{B})$  when  $\varphi(x_0, \dots, x_{n-1}) \in \mathbb{L}(\tau_M)$  and  $f_0, \dots, f_{n-1} \in M^{\mathfrak{B}}$  where  $\text{TV}$  stands for truth value and  $M^{\mathfrak{B}}/D \models \varphi[f_0/D, \dots, f_{n-1}/D]$  iff  $\text{TV}_M(\varphi(f_0, \dots, f_{n-1})) \in D$ .

2A) Abusing notation, not only  $M^{\mathfrak{B}_1} \subseteq M^{\mathfrak{B}_2}$  but  $M^{\mathfrak{B}_1}/D_1 \subseteq M^{\mathfrak{B}_2}/D_2$  when  $\mathfrak{B}_1 \triangleleft \mathfrak{B}_2, D_\ell \in \text{fil}(\mathfrak{B}_\ell)$  for  $\ell = 1, 2$  and  $D_1 = \mathfrak{B}_1 \cap D_2$ . Also  $[f_1, f_2 \in M^{\mathfrak{B}_1} \Rightarrow f_1 = f_2 \pmod{D_1} \leftrightarrow f_1 = f_2 \pmod{D_2}]$ . So for  $f \in M^{\mathfrak{B}_1}$  we identify  $f/D_1$  and  $f/D_2$ .

<sup>1</sup>In [Sh:1019] we use a  $L \subseteq \mathbb{L}_{\theta, \theta}$ ,  $\theta$  a compact cardinal and if  $\sigma > \theta$  we use a slightly different version of the definition of local and of the default values of  $\sigma$  was  $\theta$ .

<sup>2</sup>for the  $D_\ell \in \text{uf}(\mathfrak{B}_\ell)$  ultra-product, without loss of generality  $\mathfrak{B}$  is complete, then without loss of generality  $f \upharpoonright \{a_i : i < i(*)\}$  is one to one. But in general we allow  $a_i = 0_{\mathfrak{B}}$ , those are redundant but natural in 0.11(3).

3) For complete  $\mathfrak{B}$ , we say  $\langle a_n : n < \omega \rangle$  represents  $f \in \mathbb{N}^{\mathfrak{B}}$  when  $\langle a_n : n < \omega \rangle$  is a maximal antichain of  $\mathfrak{B}$  (so  $a_n = 0_{\mathfrak{B}}$  is allowed) and for some  $f' \in \mathbb{N}^{\mathfrak{B}}$  which is  $D$ -equivalent to  $f$  (see 0.11(1A)) we have  $f'(a_n) = n$ .

4) We say  $\langle (a_n, k_n) : n < \omega \rangle$  represent  $f \in \mathbb{N}^{\mathfrak{B}}$  when:

- (a) the  $k_n$  are natural numbers with no repetition
- (b)  $\langle a_n : n < \omega \rangle$  is a maximal antichain
- (c)  $f(a_n) = k_n$ .

5) If  $\mathcal{I}$  is a maximal antichain of  $\mathfrak{B}$  and  $\bar{M} = \langle M_a : a \in \mathcal{I} \rangle$  is a sequence of  $\tau$ -models, then we define  $\bar{M}^{\mathfrak{B}}$  be the set of partial functions  $f$  from  $\mathfrak{B}^+$  to  $\cup\{M_a : a \in \mathcal{I}\}$  such that for some maximal antichain  $\langle a_i : i < i(*) \rangle$  of  $\mathfrak{B}$  refining  $\mathcal{I}$  (i.e.  $(\forall i < i(*))(\exists b \in \mathcal{I})(a_i \leq_{\mathfrak{B}} b)$ ) we have:

- (a)  $(\{a_i : i < i(*)\} \subseteq \text{dom}(f) \subseteq \{b \in \mathfrak{B}^+ : b \leq_{\mathfrak{B}} a_i \text{ for some } i < i(*)\})$
- (b) if  $a \in \text{dom}(f)$  and  $a \leq a_i$  then  $f(a) = f(a_i)$
- (c) if  $a_i \leq_{\mathfrak{B}} b, b \in \mathcal{I}$  then  $f(a_i) \in M_b$ .

6) For  $\bar{M}, \mathfrak{B}, \mathcal{I}$  as above and a filter  $D$  on  $\mathfrak{B}$  we define  $\bar{M}^{\mathfrak{B}}/D$  as in part (2) replacing  $M^{\mathfrak{B}}$  there by  $\bar{M}^{\mathfrak{B}}$  here.

7) For  $\bar{M}, \mathfrak{B}, \mathcal{I}$  as above,  $\varphi = \varphi(\bar{x}) = \varphi(x_0, \dots, x_{n-1}) \in \mathbb{L}(\tau_M)$  and  $\bar{f} = \langle f_\ell : \ell < n \rangle$  where  $f_0, \dots, f_{n-1} \in \bar{M}^{\mathfrak{B}}$ , let  $\text{TV}(\varphi[\bar{f}]) = \text{TV}(\varphi[\bar{f}], \bar{M}^{\mathfrak{B}})$  be  $\sup\{a \in \mathfrak{B}^+ : \text{if } \ell < n \text{ then } a \in \text{dom}(f_\ell) \text{ and } a \leq b \in \mathcal{I} \text{ then } M_b \models \varphi[f_0(b), \dots, f_{n-1}(b)]\}$ .

8) We say  $\mathfrak{B}$  is  $\theta$ -distributive when: if  $\alpha < \theta$ ,  $\mathcal{I}_i$  is a maximal antichain of  $\mathfrak{B}$  for  $\alpha < \alpha_*$  then there is a maximal antichain of  $\mathfrak{B}$  refining every  $\mathcal{I}_i$  ( $\alpha < \alpha_*$ ); this holds, e.g. when  $\mathfrak{B} = \mathcal{P}(\lambda)$  or just there is a dense  $Y \subseteq \mathfrak{B}^+$  closed under intersection of  $< \theta$ . In this case, if  $\alpha_* < \theta$ ,  $\bar{M}, \mathfrak{B}, \mathcal{I}$  as in part (7),  $\varepsilon < \theta$ ,  $\varphi(\bar{x}_{[\varepsilon]}) \in \mathbb{L}_{\theta, \theta}$  or any logic  $f_\zeta \in \bar{M}^{\mathfrak{B}}$  for  $\zeta < \varepsilon$  the  $\text{TV}(\varphi[\bar{f}])$  is defined as in part (7).

**Definition 0.12.** Let  $\mathfrak{B}$  be a complete Boolean algebra and  $D$  a filter on  $\mathfrak{B}$ . We say that  $D$  is  $(\mu, \theta)$ -regular when for some  $(\bar{c}, \mathcal{I})$  we have:

- (a)  $\bar{c} = \langle c_\alpha : \alpha < \alpha_* \rangle$  is a maximal antichain
- (b)  $\bar{u} = \langle u_\alpha : \alpha < \alpha_* \rangle$  with  $u_\alpha \in [\mu]^{<\theta}$
- (c) if  $i < \mu$  then  $\sup\{c_\alpha : \alpha \text{ satisfies } i \in u_\alpha\} \in D$ .

**Claim 0.13.** Assume  $\mathfrak{B}$  is a complete Boolean which is  $\theta$ -distributive and  $D$  a filter on  $\mathfrak{B}$  and  $\theta = \text{cf}(\theta)$ .

1) The parallel of Los theorem holds for  $\mathbb{L}_{\theta, \theta}$  and if  $D$  is  $\lambda$ -complete even for  $\mathbb{L}_{\lambda, \theta}$  which means: if  $\bar{M} = \langle M_b : b \in \mathcal{I} \rangle$  is a sequence of  $\tau$ -models,  $\mathcal{I}$  is a maximal antichain of the complete Boolean Algebra  $\mathfrak{B}$  and  $\varepsilon < \theta$ ,  $\varphi = \varphi(\bar{x}_{[\varepsilon]}) \in \mathbb{L}_{\lambda, \theta}(\tau)$  and  $f_\zeta \in \bar{M}^{\mathfrak{B}}$  for  $\zeta < \varepsilon$  then  $M \models \varphi[\langle f_\zeta/D : \zeta < \varepsilon \rangle]$  iff  $\text{TV}(\varphi[\langle f_\zeta/D : \zeta < \varepsilon \rangle])$  belongs to  $D$ .

2) If  $D$  is  $\lambda$ -regular and  $M, N$  are  $\mathbb{L}_{\theta, \theta}$ -equivalent then  $M^{\mathfrak{B}}/D, N^{\mathfrak{B}}/D$  are  $\mathbb{L}_{\lambda^+, \theta}$ -equivalent.

**Definition 0.14.** 1) Assume  $\Delta_\ell$  is a set of atomic formulas in  $\mathbb{L}(\tau(T_\ell))$ . Then we say  $(T_1, \Delta_1) \leq_{\lambda, \theta}^{\text{rp}} (T_2, \Delta_2)$  when: if  $D$  is a  $\lambda$ -regular filter on  $\lambda$  and  $M_\ell$  is a  $(\lambda^+$ -saturated) model of  $T_\ell$  for  $\ell = 1, 2$  and  $M_2^\lambda/D$  is  $(\lambda, \theta, \Delta_2)$ -saturated then  $M_1^\lambda/D$  is  $(\lambda, \theta, \Delta_1)$ -saturated.

2) For general  $\Delta_1, \Delta_2$  we define  $(T_1, \Delta_1) \leq_{\lambda, \theta}^{\text{rp}} (T_2, \Delta_2)$  as meaning  $(T_1^+, \Delta_1^+) \leq_{\lambda, \theta}^{\text{rp}} (T_2^+, \Delta_2^+)$  where (as Morley [Mor65] does):

- $T_\ell^+ = T_\ell \cup \{(\forall \bar{x})(\varphi(\bar{x}) \equiv P_\varphi(\bar{x})) : \varphi(\bar{x}) \in \Delta_\ell\}$  with  $\langle P_\varphi^\ell : \varphi \in \Delta_\ell \rangle$  new pairwise distinct predicates with suitable number of places
- $\Delta_\ell^+ = \{P_\varphi^\ell(\bar{x}_\varphi) : \varphi \in \Delta_\ell\}$ .

3) In (2),  $T_1 \leq_{\lambda, \theta}^{\text{rp}} T_2$  means  $\Delta_\ell =$  the set of atomic  $\mathbb{L}_{\theta, \theta}(\tau_{T_\ell})$ -formulas.

## § 1. AXIOMATIZING [Sh:c, Ch.VI,2.6]

- Definition 1.1.** 1) For partial order  $\mathcal{T} = (\mathcal{T}, \leq_{\mathcal{T}})$  let  $\mathbf{p}_{\mathcal{T}} = \mathbf{p}(\mathcal{T})$  be  $\min\{\kappa_1 + \kappa_2 : (\kappa_1, \kappa_2) \in \mathcal{C}_{\mathcal{T}}\}$  and  $\mathbf{p}_{\theta}(\mathcal{T}) = \min\{\kappa_1 + \kappa_2 : (\kappa_1, \kappa_2) \in \mathcal{C}_{\mathcal{T}, \theta}\}$ ; where:  
 2)  $\mathcal{C}_{\mathcal{T}} = \mathcal{C}(\mathcal{T}) = C_{\mathcal{T}, \theta}$  where  $\mathcal{C}_{\mathcal{T}, \theta} = \{(\kappa_1, \kappa_2) : \text{the partial order } \mathcal{T}, \kappa \geq \aleph_0 \text{ has a } (\kappa_1, \kappa_2)\text{-cut and } \kappa_1 \geq \theta, \kappa_2 \geq \aleph_0\}$ .  
 3) For a partial order  $\mathcal{T}$  let  $\mathbf{t}_{\mathcal{T}} = \mathbf{t}(\mathcal{T})$  be the minimal  $\kappa \geq \aleph_0$  such that there is a  $<_{\mathcal{T}}$ -increasing sequence of length  $\kappa$  with no  $<_{\mathcal{T}}$ -upper bound.  
 4) Let  $\mathbf{p}_{\mathcal{T}}^* = \mathbf{p}^*(\mathcal{T})$  be  $\min\{\mathbf{t}_{\mathcal{T}}, \mathbf{p}_{\mathcal{T}}\}$ .  
 5)  $\mathbf{p}_{\text{sym}}^*(\mathcal{T}) = \min\{\kappa : (\kappa, \kappa) \in \mathcal{C}(\mathcal{T})\}$ .  
 6) In Definition 1.2 below let  $\mathbf{t}_{\mathbf{r}} = \mathbf{t}_{\mathcal{T}_{\mathbf{r}}}$ ,  $\mathbf{p}_{\mathbf{r}} = \mathbf{p}_{\mathcal{T}_{\mathbf{r}}, \theta_{\mathbf{r}}}$ .

**Definition 1.2.** 1) For  $\iota = 1, 2$  we say  $\mathbf{r}$  or  $(M, \Delta)$  is a  $(\theta, \iota)$ -realization<sup>3</sup> spectrum problem, in short  $(\theta, \iota)$ -RSP or  $(\theta, \iota)$ -1-RSP when  $\mathbf{r}$  consists of (if  $\iota = 2$  we may omit it, similarly if  $\theta = \aleph_0$ ; we may omit  $\Delta$  and write  $M$  when  $\Delta$  is the set of atomic formulas in  $\mathbb{L}(\tau_{N_M})$ , see below, so  $M$  below =  $M_{\mathbf{r}}$ , etc.):

- (a)  $M$  a model
- (b) for the relations  $\mathcal{T} = \mathcal{T}^M, \leq_{\mathcal{T}} = \leq_{\mathcal{T}}^M$  of  $M$  (i.e.  $\mathcal{T}, \leq_{\mathcal{T}}$  are predicates from  $\tau_M$ ) we have  $\mathcal{T} = (\mathcal{T}, \leq_{\mathcal{T}})$  a partial order (so definable in  $M$ ) with root  $c^M = \text{rt}(\mathcal{T})$ , so  $c \in \tau_M$  is an individual constant and  $t \in \mathcal{T} \Rightarrow \text{rt}(\mathcal{T}) \leq_{\mathcal{T}} t$ ; as in other cases we may write  $\mathcal{T}_{\mathbf{r}}, \leq_{\mathbf{r}}$  for  $\mathcal{T}, \leq_{\mathcal{T}}$ ; we do not require  $\mathcal{T}$  to be a tree; but do require  $t \in \mathcal{T} \Rightarrow t \leq_{\mathcal{T}} t$
- (c) a model  $N = N_{\mathbf{r}} = N_M$  with universe  $P^M, \tau(N) \subseteq \tau(M)$  such that
  - $Q \in \tau_N \Rightarrow Q^M = Q^N$
  - $F \in \tau_N \Rightarrow F^N = F^M$ , (we understand  $F^M, F^N$  to be partial functions),  
 so every  $\varphi \in \mathbb{L}(\tau_N)$  can be interpreted as  $\varphi^{[*]} \in \mathbb{L}(\tau_M)$ , all variables varying on  $P$  (include quantification); we may forget the  $[*]$ .
- (d) the cardinal  $\theta$  and  $\Delta \subseteq \{\varphi : \varphi = \varphi(x, \bar{y}) \in \mathbb{L}_{\theta, \theta}(\tau_N)\}$  which is closed under conjunctions meaning: if  $\varphi_{\ell}(x, \bar{y}_{\ell}) \in \Delta$  for  $\ell = 1, 2$  then  $\varphi(x, \bar{y}'_1, \bar{y}''_2) = \varphi_1(x, \bar{y}'_1) \wedge \varphi_2(x, \bar{y}''_2) \in \Delta$
- (e)  $R^M \subseteq |N| \times \mathcal{T}^M$  so a two-place relation; and let  $R_t^M = \{b : bR^M t\}$  for  $t \in \mathcal{T}^M$
- (f)  $|N| \times \{\text{rt}_{\mathcal{T}}\} \subseteq R^M$ , i.e.  $R_{\text{rt}(\mathcal{T})}^M = |N|$
- (g) if  $s \leq_{\mathcal{T}} t$  then  $a \in N \wedge aRt \Rightarrow aRs$ , i.e.  $R_s^M \supseteq R_t^M$
- (h)  $t \in \mathcal{T} \Rightarrow R_t^M \neq \emptyset$
- (i) if  $s \in \mathcal{T}, \varphi(x, \bar{a}) \in \Delta(N) := \{\varphi(x, \bar{a}) : \varphi(x, \bar{y}) \in \Delta \text{ and } \bar{a} \in {}^{\ell g(\bar{y})}N\}$  and for some  $b \in R_s^M, N \models \varphi[b, \bar{a}]$  then there is  $t \in \mathcal{T}$  such that  $s \leq_{\mathcal{T}} t$  and  $R_t^M = \{b \in R_s^M : N \models \varphi[b, \bar{a}]\}$
- (i)<sup>+</sup> if  $\iota = 1$  like clause (i) but<sup>4</sup> moreover  $t = F_{\varphi, 1}^M(s, \bar{a})$  where  $F_{\varphi, 1}^M : \mathcal{T}_{\mathbf{r}} \times {}^{\ell g(\bar{y})}(P^M) \rightarrow \mathcal{T}_{\mathbf{r}}$
- (j) if  $t \in \mathcal{T}_{\mathbf{r}}$  and  $\varphi(x, \bar{a}) \in \Delta(N)$  and  $\varphi(N, \bar{a}) \neq \emptyset$  then

<sup>3</sup>When  $P$  and  $\tau_N$  (hence  $N$ ) are understood from the context we may omit them

<sup>4</sup>We may not add a function, maybe it matters when we try to build  $\mathbf{r}$  with  $\text{Th}(M_{\mathbf{r}})$ -nice first order



- ( $\alpha$ )  $s = F_{\varphi,2}^M(t, \bar{a})$  is such that  $R_s^M \cap \varphi(N, \bar{a}) \neq \emptyset$  and  $s \leq_{\mathcal{T}} t$
- ( $\beta$ ) if  $s = F_{\varphi,2}^M(t, \bar{a})$ ,  $s_1 \leq_{\mathcal{T}} t$  and  $R_{s_1}^M \cap \varphi(N, \bar{a}) \neq \emptyset$  then  $s_1 \leq_{\mathcal{T}} s$
- ( $k$ ) if  $\theta > \aleph_0$  then in  $(\mathcal{T}, \leq_{\mathcal{T}})$  any increasing chain of length  $< \theta$  which has an upper bound has a  $\leq_{\mathcal{T}}$ -lub.

*Remark 1.3.* We may consider adding:  $S^M$  a being successor, (but this is not Horn), i.e.:

- ( $l$ ) if  $\iota = 1$  we also have  $S^M = \{(a, b) : B \text{ is a } \leq_{\mathcal{T}}\text{-successor of } a \text{ such that}$ 
  - ( $\alpha$ ) if  $a \leq b \wedge a \neq b$  then for some  $c$ ,  $S(a, c) \wedge c \leq b$
  - ( $\beta$ ) if  $b \in \mathcal{T} \setminus \{\text{rt}_{\mathcal{T}}\}$  then for some unique  $a$  we have  $S^M(a, b)$
  - ( $\gamma$ )  $S(a, b) \Rightarrow a \leq b$
  - ( $\delta$ )  $S(a, b_1) \wedge S(a, b_2) \wedge b_1 \neq b_2 \Rightarrow \neg(b_1 \leq b_2)$
  - ( $\varepsilon$ ) in clause (j) we can add  $S^M(s, t)$ .

*Remark 1.4.* Presently, it may be that  $a \leq_{\mathcal{T}} b \leq_{\mathcal{T}} a$  but  $a \neq b$ . Not a disaster to forbid but no reason.

How does this axiomatize realizations of types?

**Claim/Definition 1.5.** Let  $\iota = \{1, 2\}$ ,  $\theta$  is  $\aleph_0$  or just a regular cardinal.

- 1) For any model  $N$  and  $\Delta \subseteq \{\varphi = \varphi(x, \bar{y}) \in \mathbb{L}_{\theta, \theta}(\tau_T)\}$  closed under conjunctions of  $< \theta$ , the canonical  $(\theta, \iota)$ -RSP,  $\mathbf{r} = \mathbf{r}_{N, \Delta}^{\theta}$  defined below is indeed a  $\theta$ -RSP.
- 2)  $\mathbf{r} = \mathbf{r}_{N, \Delta}^{\theta}$  (if  $\theta = \aleph_0$  we may omit it) is defined by:

- (a)  $\Delta_{\mathbf{r}} = \Delta$ ,  $N_{\mathbf{r}} = N$  and  $\theta_{\mathbf{r}} = \theta$
- (b)  $\mathcal{T}_{\mathbf{r}} = \{\langle \varphi_{\varepsilon}(x, \bar{a}_{\varepsilon}) : \varepsilon < \zeta \rangle : \zeta < \theta \text{ and for every } \varepsilon < \zeta \text{ we have } \varphi_{\varepsilon}(x, \bar{a}_{\varepsilon}) \in \Delta(N) \text{ and } N \models (\exists x)(\bigwedge_{\varepsilon < \zeta} \varphi_{\varepsilon}(x, \bar{a}_{\varepsilon}))\}$
- (c)  $\leq_{\mathbf{r}}$  being the initial segment relation on  $\mathcal{T}_{\mathbf{r}}$
- (d)  $M = M_{\mathbf{r}}$  is the model with universe  $\mathcal{T}_{\mathbf{r}} \cup |N|$ ; without loss of generality  $\mathcal{T}_{\mathbf{r}} \cap |N| = \emptyset$ , with the relations and functions of  $N$ ,  $\mathcal{T}_{\mathbf{r}}$ ,  $\leq_{\mathbf{r}}$  and
  - $P^M = |N|$
  - $c^M = \langle \rangle \in \mathcal{T}_{\mathbf{r}}$
  - $R^M = \{(b, t) : a \in N, t = \langle \varphi_{t, \varepsilon}(x, \bar{a}_{t, \varepsilon}) : \varepsilon < \zeta_t \rangle \in \mathcal{T} \text{ and } N \models \varphi_{t, \ell}(b, \bar{a}_{t, \varepsilon}) \text{ for every } \varepsilon < \zeta_t\}$
  - $F_{\varphi, 2}^M$  as in Definition 1.2(j)
  - if  $\iota = 1$  then  $F_{\varphi, 1}^M$  is as in Definition 1.2(i)<sup>+</sup>.

*Remark 1.6.* If we adopt 1.3 it is natural to add:

- (e) for  $\iota = 1$ ,  $S^M = \{(\bar{\varphi}_1, \bar{\varphi}_2) : \bar{\varphi}_2 = \bar{\varphi}_1 \hat{\ } \langle \varphi(x, \bar{a}) \rangle \in \mathcal{T}_{\mathbf{r}} \text{ for some } \varphi(x, \bar{a}) \in \Delta(N)\}$ .

*Proof.* Obvious. □<sub>1.5</sub>

**Main Claim 1.7.** 1) Assume  $\mathbf{r}$  is an RSP. If  $\kappa = \min\{\mathfrak{t}_{\mathbf{r}}, \mathfrak{p}_{\mathbf{r}}\}$  then the model  $N$  is  $(\kappa, 1, \Delta_{\mathbf{r}})$ -saturated, i.e.

- ⊕ if  $p(x) \subseteq \Delta_{\mathbf{r}}(N_{\mathbf{r}})$  is finitely satisfiable in  $N_{\mathbf{r}}$  (= is a type in  $N_{\mathbf{r}}$ ) of cardinality  $< \kappa$  then  $p$  is realized in  $N_{\mathbf{r}}$ .

2) If  $\theta > \aleph_0$  and  $\mathbf{r}$  is a  $\theta$ -RSP, then  $N_{\mathbf{r}}$  is  $(\kappa, 1, \Delta_{\mathbf{r}})$ -saturated where  $\kappa = \min\{\mathfrak{t}_{\mathbf{r}}, \mathfrak{p}_{\mathbf{r}}\}$  recalling 1.1(6), i.e.  $\mathfrak{p}_{\mathbf{r}} = \mathfrak{p}_{\mathcal{T}, \theta}$ .

3) If  $\theta > \aleph_0$ ,  $\mathbf{r}$  is a  $\theta$ -RSP satisfying  $(k)^+$  below then  $N_{\mathbf{r}}$  is  $(\mathfrak{t}_{\mathbf{r}}, 1, \Delta_{\mathbf{r}})$ -saturated when:

$(k)^+$  in  $(\mathcal{T}, \leq_{\mathcal{T}})$  any increasing chain which has an upper bound, has a  $\leq_{\mathcal{T}}$ -lub.

*Proof.* This is an abstract version of [Sh:a, Ch.VI,2.6] = [Sh:c, Ch.VI,2.6]; recall that [Sh:a, Ch.VI,2.7] translates trees to linear orders.

1) Let  $N = N_{\mathbf{r}}, \Delta = \Delta_{\mathbf{r}}$ , etc.

Let  $p$  be a  $(\Delta, 1)$ -type in  $N$  of cardinality  $< \kappa$ . Without loss of generality  $p$  is infinite and closed under conjunctions.

So let

$(*)_1$   $\alpha_* < \kappa, p = \{\varphi_{\alpha}(x, \bar{a}_{\alpha}) : \alpha < \alpha_*\} \subseteq \Delta(N), p$  is finitely satisfiable in  $N$ .

We shall try to choose  $t_{\alpha}$  by induction on  $\alpha \leq \alpha_*$  such that

- $(*)_2$  (a)  $t_{\alpha} \in \mathcal{T}$  and  $\beta < \alpha \Rightarrow t_{\beta} \leq_{\mathcal{T}} t_{\alpha}$
- (b) if  $\beta < \alpha_*$  then there is  $b \in R_{t_{\alpha}}^M$  such that  $N \models \varphi_{\beta}[b, \bar{a}_{\beta}]$
- (c) if  $\beta < \alpha$  then  $b \in R_{t_{\alpha}}^M \Rightarrow N \models \varphi_{\beta}[b, \bar{a}_{\alpha}]$ .

If we succeed, this is enough because if  $t = t_{\alpha_*}$  is well defined then  $R_t^M \neq \emptyset$  by Definition 1.2(h) and any  $b \in R_t^M$  realizes the type by  $(*)_2(c)$  and Definition 1.2(h). Why can we carry the definition?

Case 1:  $\alpha = 0$

Let  $t_{\alpha} = \text{rt}_{\mathcal{T}}$ , hence  $R_{t_{\alpha}}^M = |N|$  by Definition 1.2(f). Now clause (a) of  $(*)_2$  holds as  $t_{\alpha} \in \mathcal{T}_{\mathbf{r}}$  and there is no  $\beta < \alpha$ . Also clause (b) of  $(*)_2$  holds because  $p$  is a type and  $R_{\text{rt}(\mathcal{T})}^M = |N_{\mathbf{r}}|$  by Definition 1.2(h).

Lastly, clause (c) of  $(*)_2$  holds trivially.

Case 2:  $\alpha = \beta + 1$

If  $\iota = 1$  let  $t = F_{\varphi_{\beta}, 1}^M(t_{\beta}, \bar{a}_{\beta})$  and see clause (i)<sup>+</sup> of Definition 1.2. If  $\iota = 2$  use clause (i) of the definition recalling  $p$  is closed under conjunctions.

Case 3:  $\alpha$  a limit ordinal

As  $\mathfrak{t}_{\mathcal{T}} \geq \kappa > \alpha_*$  by the claim's assumption (on  $\mathfrak{t}_{\mathcal{T}}$ , see Definition 1.1(2)) necessarily there is  $s \in \mathcal{T}$  such that  $\beta < \alpha \Rightarrow t_{\alpha} \leq_{\mathcal{T}} s$ . We now try to choose  $s_i$  by induction on  $i \leq \alpha_*$  such that

- $(*)_{2.1}$  (a)  $s_i \in \mathcal{T}$
- (b)  $\beta < \alpha \Rightarrow t_{\beta} \leq_{\mathcal{T}} s_i$
- (c)  $j < i \Rightarrow s_i \leq_{\mathcal{T}} s_j$
- (d) if  $i = j + 1$  then  $R_{s_i}^M$  is not disjoint to  $\varphi_j(N, \bar{a}_j)$ .

If we succeed, then  $s_{\alpha_*}$  satisfies all the demands on  $t_{\alpha}$  (e.g.  $(*)_2(b)$  holds by Definition 1.2(g) and  $(*)_{2.1}(d)$ ), so we have just to carry the induction for  $\alpha$ . Now if  $i = 0$  clearly  $s_0 = s$  it as required. If  $i = j + 1$  let  $s_i = F_{\varphi_j, 2}^M(s_j, \bar{a}_j)$ , by Definition 1.2(j) it is as required. For  $i$  a limit ordinal use  $\kappa \leq \mathfrak{p}_{\mathcal{T}}$  hence to carry the induction on  $i$  so finish case 3.

So we succeed to carry the induction on  $\alpha$  hence (as said after  $(*)_2$ ) get the desired conclusion.

2) Similar, except concerning case 3. Note that without loss of generality  $\theta > \aleph_0$  by part (1).

Case 3A:  $\alpha$  is a limit ordinal of cofinality  $\geq \theta$

As in the proof of part (1).

Case 3B:  $\alpha$  is a limit ordinal of cofinality  $< \theta$

Again there is an upper bound  $s$  of  $\{t_\beta : \beta < \alpha\}$ . Now by clause (k) of Definition 1.2, without loss of generality  $s$  is a  $<\mathcal{T}$ -lub of  $\{t_\beta : \beta < \alpha\}$ . So easily for every  $i < \alpha_*$ ,  $F_{\varphi_i,2}^N(s, \bar{a}_i)$  is  $\geq t_\beta$  for  $\beta < \alpha$  hence is equal to  $s$ , so  $s_\alpha := s$  is as required.

3) Similarly.

□<sub>1.7</sub>

**Discussion 1.8.** 1) What about “ $(\lambda^+, n, \Delta)$ -saturation”? We can repeat the same analysis or we can change the models to code  $n$ -tuples. More generally, replacing  $\varphi(\bar{x}_{[\varepsilon]}, \bar{y})$  by  $\varphi(\langle F_\zeta(x) : \zeta < \varepsilon \rangle, \bar{y})$ , using  $F_\zeta \in \tau_M$  (though not necessarily  $F_\zeta \in \tau_{N_r}$ ), so we can allow infinite  $\varepsilon$ .

2) Hence the same is true for  $(\lambda^+, \aleph_0, \Delta)$ -saturation, e.g.  $\lambda^+$ -saturated by an assumption.

## § 2. APPLYING THE AXIOMATIZED FRAME

Of course, (note that we do not require  $D$  to be regular); if it is  $\lambda$ -regular, then we can reduce  $\mu$  if  $D$  is  $\lambda$ -regular,  $\lambda \geq |\tau_N|$  then we can replace  $(^{\omega>} \mu, \triangleleft)$  by  $(^{\omega>} \omega, \triangleleft)$ , see 0.13(2).

**Conclusion 2.1.** 1) If  $D$  is an ultrafilter on a set  $I, N$  a model,  $\mu = \|N\| + \|\tau_N\|$  and  $(^{\omega>} \mu, \triangleleft)^\lambda / D$  is  $(\lambda^+, \text{atomically})$ -saturated then  $N^+ / D$  is  $\lambda^+$ -saturated.

2) Instead “ $(^{\omega>} \mu, \trianglelefteq)^I / D$  is  $(\lambda^+, 1, \text{atomically})$ -saturated” we can demand  $J^I / D$  is  $(\lambda^+, 1, \text{atomic})$ -saturated where  $J$  is the linear order with set of elements  $\{-1, 1\} \times ^{\omega>} \mu$  ordered by  $(\iota_1, \eta_1) < (\iota_2, \eta_2)$  iff  $\iota_1 < \iota_2$  or  $\iota_1 = -1 = \iota_2 \wedge \eta_1 <_{\text{lex}} \eta_2$  or  $\iota_1 = -1 = \iota_2 \wedge \eta_2 <_{\text{lex}} \eta_1$ .

*Proof.* 1) Let  $N_1 = N$ . As  $D$  is an ultrafilter without loss of generality  $\text{Th}(N_1)$  has elimination of quantifiers. Let  $\Delta = \mathbb{L}(\tau_N)$ , by 1.5  $\mathbf{r}_1 := \mathbf{r}_{N_1, \Delta}$  is an RSP. Let  $N_2 = N_1^I / D = M_1 = r_{\mathbf{r}_1}, M_2 = M_1^I / D$  and let  $\mathbf{r}_2$  be the RSP( $M_2, \Delta$ ). Clearly  $\mathbf{r}_2$  is an RSP as the demands in 1.2 are first order (see more in 2.2).

Now

$$(*)_1 \quad \mathcal{T}_{\mathbf{r}_1} \cong (^{\omega>} \mu, \triangleleft).$$

[Why? See 1.5(2).]

$$(*)_2 \quad \mathcal{T}_{\mathbf{r}_2} = (\mathcal{T}_{\mathbf{r}_1})^I / D \text{ is } (\lambda^+, \text{atomically})\text{-saturated.}$$

[Why? By an assumption.]

$$(*)_3 \quad \mathfrak{t}(\mathcal{T}_{\mathbf{r}_1}), \mathfrak{p}(\mathcal{T}_{\mathbf{r}_2}) \geq \lambda^+.$$

[Why? Follows by  $(*)_2$ .]

Hence by 1.7,  $N_2$  is  $(\lambda^+, 1, 1, \Delta)$ -saturated which means  $N_2 = (N_1)^I / D$  is  $\lambda^+$ -saturated.

2) Easy (or see [Sh:a, Ch.VI,2.7] or see [Sh:14]).

□<sub>2.1</sub>

To apply the criterion of the Main Claim 1.7 to reduced products we need:

**Claim 2.2.** If  $\Delta$  is the set of conjunctions of atomic formulas (no negation!) in  $\mathbb{L}(\tau_0)$  and  $\tau = \{\mathcal{T}, \leq_{\mathcal{T}}, R, P, c\} \cup \{F_{\varphi, \ell} : \varphi \in \Delta \text{ and } \ell = 2 \text{ or } \ell = 1 \text{ if relevant}\} \cup \tau_0$  (recall  $c$  is  $\text{rt}_{\mathcal{T}}$ ), then there is set  $T$  of Horn sentences from  $\mathbb{L}(\tau)$  such that for every  $\tau$ -model  $M$

- $(M, \Delta)$  is a RSP (i.e. 2-RSP) iff  $M \models T$ .

*Proof.* Consider Definition 1.2. For each clause we consider the sentences expressing the clauses there.

Clause (a): Obvious

Clause (b): Clearly the following are Horn:

- $x \leq_{\mathcal{T}} y \rightarrow \mathcal{T}(x), x \leq_{\mathcal{T}} y \rightarrow \mathcal{T}(y)$
- $x \leq_{\mathcal{T}} y \wedge y \leq_{\mathcal{T}} z \rightarrow x \leq_{\mathcal{T}} z,$
- $\mathcal{T}(\text{rt}_{\mathcal{T}})$  and  $\mathcal{T}(s) \rightarrow \text{rt}_{\mathcal{T}} \leq s$
- $\mathcal{T}(x) \rightarrow x \leq_{\mathcal{T}} x.$

Note that  $(\mathcal{T}, \leq_{\mathcal{T}})$  being a tree is not a Horn sentence but is not required.

Clause (c):

- $Q(x_0, \dots, x_{n(Q)-1}) \rightarrow P(x_\ell)$  when  $Q$  is an  $n(Q)$ -place predicate from  $\tau(N)$  and  $\ell < n(Q)$ ; clearly it is Horn
- for any  $n$ -place function symbol  $F \in \tau_0$  the sentence:  $P(x_0) \wedge \dots \wedge P(x_{n-1}) \rightarrow P(F(x_0, \dots, x_{n-1}))$  and  $y = F(x_0, \dots, x_{n-1}) \rightarrow P(x_\ell)$ .

Clause (d): nothing to prove - see the present claim assumption on  $\Delta$ .

Clause (e):  $yRs \rightarrow \mathcal{T}(s), yRs \rightarrow P(y)$  are Horn.

Clause (f):  $P(x) \rightarrow xR(\text{rt}_{\mathcal{T}})$  is Horn.

Clause (g):  $s \leq_{\mathcal{T}} t \wedge xRt \rightarrow xRs$  is Horn.

Clause (h):  $(\forall t)(\exists x)(\mathcal{T}(t) \rightarrow xRt)$  is Horn.

Clause (i): Let  $\varphi(x, \bar{y}) \in \Delta$ .

First assume  $\iota = 1$ . Note the following are Horn: for any  $\varphi(x, \bar{y}) \in \Delta$

- $\mathcal{T}(s) \wedge xRs \wedge \varphi(x, \bar{y}) \wedge \bigwedge_{\ell < \ell g(\bar{y})} P(y_\ell) \wedge t = F_{\varphi,1}(s, \bar{y}) \rightarrow \mathcal{T}(t) \wedge s \leq_{\mathcal{T}} t$
- $\mathcal{T}(s) \wedge xRs \wedge \varphi(x, \bar{y}) \wedge \bigwedge_{\ell < \ell g(\bar{y})} P(y_\ell) \wedge t = F_{\varphi,1}(s, \bar{y}) \rightarrow xRt$
- $\mathcal{T}(s) \wedge x'R s \wedge x'R F_{\varphi,1}(s, \bar{y}) \rightarrow \varphi(x', \bar{y})$ .

This suffices. The proof when  $\iota = 2$  is similar.

Clause (j): Similarly but we give details.

Let  $\varphi = \varphi(x, \bar{y}) \in \Delta$ , so the following are Horn:

- $\varphi(x_1, \bar{y}) \wedge P(x_1) \wedge \bigwedge_{\ell < \ell g(\bar{y})} P(y_\ell) \wedge s = F_{\varphi,2}(t, \bar{y}) \rightarrow s \leq_{\mathcal{T}} t$
- $\varphi(x_1, \bar{y}) \wedge P(x_1) \wedge \bigwedge_{\ell < \ell g(\bar{y})} P(y_\ell) \wedge s = F_{\varphi,2}(t, \bar{y}) \rightarrow (\exists x)(xRs \wedge \varphi(x, \bar{y}))$
- $P(x) \wedge \bigwedge_{\ell < \ell g(\bar{y})} P(y_\ell) \wedge s = F_{\varphi,2}(t, \bar{y}) \wedge z \leq_{\mathcal{T}} t \wedge xRz \wedge \varphi(x, \bar{y}) \rightarrow z \leq_{\mathcal{T}} s$ .

Clause (k): As  $\theta = \aleph_0$  this is empty.

This suffices. □<sub>2.2</sub>

**Claim 2.3.** *Also for  $\theta > \aleph_0$  (see 1.2(2)) Claim 2.2 holds but some of the formulas are in  $\mathbb{L}_{\theta, \theta}$ .*

*Proof.* Clause (k): When  $\theta > \aleph_0$ .

Should be clear because for each limit ordinal  $\delta < \kappa$ , the sentences  $\psi_\delta = (\forall x_0, \dots, x_\alpha, \dots, x_\delta)(\exists y)(\forall z)((\bigwedge_{\alpha < \beta < \delta} x_\alpha \leq_{\mathcal{T}} x_\beta \leq_{\mathcal{T}} y \leq_{\mathcal{T}} x_\delta) \wedge (\bigwedge_{\alpha < \beta < \delta} x_\alpha \leq_{\mathcal{T}} x_\beta \leq_{\mathcal{T}} z \leq_{\mathcal{T}} y \leq_{\mathcal{T}} x_\delta) \rightarrow y = z)$  is a Horn sentence and it expresses “any  $\leq_{\mathcal{T}}$ -increasing chain of length  $\delta$  has a  $\leq$ -lub”. □<sub>2.3</sub>

**Conclusion 2.4.** 1) Assume

- (a)  $D$  be a filter on  $I$
- (b)  $N$  a model,  $\lambda = \|N\| + |\tau_N|$ ,  $\Delta$  the set of atomic formulas (in  $\mathbb{L}(\tau_N)$ )

- (c)  $\mathcal{T} = (\mathcal{T}, \leq_{\mathcal{T}}) := (\omega^{>\lambda}, \trianglelefteq)^I / D$
- (d)  $\kappa = \mathfrak{p}_{\mathcal{T}}^* = \min\{\mathfrak{t}_{\mathcal{T}}, \mathfrak{p}_{\mathcal{T}_1, \theta}\}$ , see Definition 1.1(6).

Then the reduced power  $N^I / D$  is  $(\kappa, 1, \Delta)$ -saturated.

2) Assume

- (a)  $D$  is a  $\theta$ -complete filter on  $I$ ,  $\theta = \text{cf}(\theta) > \aleph_0$
- (b)  $N$  is  $(\theta, \Delta)$ -saturated,  $\Delta$  a set of atomic formulas
- (c)  $\mathcal{T}_1 := (\theta^{>\lambda}, \trianglelefteq)^I / D$
- (d)  $\kappa = \min\{\mathfrak{t}_{\mathcal{T}_1}, \mathfrak{p}_{\mathcal{T}_1, \theta}\}$ .

Then  $N^I / D$  is  $\kappa$ -saturated.

3) We can above replace  $N^I / D$  by  $N^{\mathfrak{B}} / D$  where  $D$  is a filter on the complete Boolean Algebra  $\mathfrak{B}$  which has  $(\infty, \theta)$ -distributivity when  $\theta > \aleph_0$ .

*Proof.* 1) Let  $\theta = \aleph_0$  and  $\mathbf{r}_0 = (M_0, \Delta)$  be  $\mathbf{r}_{N, \Delta}^\theta$  from 1.5, so  $\theta_{\mathbf{r}_0} = \theta$ .

By Claim 1.5,  $M_0$  is an RSP hence by Claim 2.2 also  $M = M_0^I / D$  is an RSP. Now apply the Main Claim 1.7(1).

2) Similarly using 1.7(2).

3) Similarly. □<sub>2.4</sub>

*Remark 2.5.* 1) No harm in assuming  $\Delta = \{Q(\bar{y}) : Q \text{ a predicate}\}$ . Note that allowing bigger  $\Delta$  is problematic except in trivial cases ( $\varphi$  and  $\neg\varphi$  are equivalent to Horn formulas), see proof of clauses (i),(j) of Definition 1.2.

2) Using 2.4(1) above, if  $D$  is an ultrafilter, not surprisingly we get [Sh:c, Ch.VI,2.6], i.e. the theory of dense linear orders is  $\triangleleft$ -maximal (well, using the translation from dense linear orders to trees in 2.1(2) equivalently [Sh:c, Ch.VI,2.7]). The new point is that 2.4 does this also for reduced powers, i.e. for  $D$  a filter.

3) So a natural question is can we replace the strict property by  $\text{SOP}_2$ . But in reduced power we have also non-peculiar cuts, see §4.

4) Why is the reduced power of a tree not necessarily a tree? Let  $\eta_1 \triangleleft \eta_2 \triangleleft \eta_3 \in \omega^{>\omega}$  and let  $A_1, A_2 \in I^+$  be disjoint and define  $f_\ell : I \rightarrow \omega^{>\omega}$  for  $\ell = 1, 2, 3$  by:

- $f_3(s) = \eta_3$  for  $s \in I$
- $f_2(s)$  is  $\eta_2$  if  $s \in A_2$  and  $\eta_0$  otherwise
- $f_1(s)$  is  $\eta_2$  if  $s \in A_1$  and  $\eta_0$  otherwise.

Clearly in  $N = M^I / D$  we have:

- $f_1 / D \triangleleft f_3 / D$
- $f_2 / D \triangleleft f_3 / D$
- $\neg(f_1 / D \triangleleft f_2 / D)$
- $\neg(f_2 / D \triangleleft f_1 / D)$
- $\neg(f_1 / D = f_2 / D)$ .

**Conclusion 2.6.**  $N^I / D$  is  $(\kappa, 1, \Delta_1)$ -saturated and  $\kappa \geq \theta$  when:

- (\*) (a)  $D \in \text{fil}_\theta(I)$ , i.e. is a  $\theta$ -complete filter on  $I$
- (b)  $\Delta \subseteq \{\varphi : \varphi(x, \bar{y}) \in \mathbb{L}_{\theta, \theta}(\tau_M) \text{ is atomic (hence } \in \mathbb{L}(\tau_M))\}$
- (c)  $\Delta_1 = \text{cl}_{<\theta}(\Delta)$  = the closure of  $\Delta$  under conjunction of  $< \theta$  formulas

- (d)  $N$  is  $(\theta, \Delta)$ -saturated, i.e. if  $p(x) \subseteq \Delta(M)$  has cardinality  $< \theta$  and is finitely satisfiable in  $M$  then  $p$  is realized in  $N$
- (e)  $\kappa = \min\{\mathfrak{p}_{\mathcal{T}}, \mathfrak{t}_{\mathcal{T}, \theta}\}$  where  $\mathcal{T} = (\theta^{>\lambda}, \leq)^I/D$  and  $\lambda = \theta^{>}(\|M\| + |\Delta|)$ .

*Proof.* Let  $\mathbf{r} = \mathbf{r}_{N, \Delta_1}^\theta$  recalling Definition 1.5 and  $M_0 = M_{\mathbf{r}}$ .

Now apply 1.7(2) noting that:

- (\*)<sub>1</sub>  $M_1 = M_0^I/D$  satisfies: every set of  $< \theta$  formulas from  $\Delta(M)$  which is finitely satisfiable in  $N_1$  is realized in  $N_1$ .

[Why? Let  $\langle \varphi_\alpha(x, f_{\alpha,0}/D, \dots, f_{\alpha, n(\alpha)-1}/D) : \alpha < \alpha_* \rangle$  be finitely satisfiable in  $N_1$  and  $\alpha_* < \theta, \alpha < \alpha_* \Rightarrow \varphi_\alpha \in \Delta$ . For every finite  $u \subseteq \alpha_*$  we have  $N_1 \models (\exists x)(\bigwedge_{\alpha \in u} \varphi_\alpha(x, f_{\alpha,0}/D, \dots))$  hence the set

$$I_u := \{s \in I : N_1 \models (\exists x) \bigwedge_{\alpha \in u} \varphi_\alpha(x, f_{\alpha,0}(s), \dots, f_{\alpha, n(\alpha)-1}(s))\}$$

belongs to  $D$ . But  $D$  is  $\theta$ -complete, hence  $I_* = \bigcap \{I_u : u \subseteq \alpha_* \text{ is finite}\}$  belongs to  $D$ . Now for each  $s \in I_*$ , the set  $p_s := \{\varphi_\alpha(x, f_{\alpha,0}(s), \dots, f_{\alpha, n(\alpha)-1}(s)) : \alpha < \alpha_*\}$  is finitely satisfiable in  $N$ , hence is realized by some  $a_s \in N$ . Let  $g \in {}^I N$  be such that  $s \in I_* \Rightarrow g(s) = a_s$ ; clearly  $g/D$  realizes  $p$ , so we are done.]

Similarly

- (\*)<sub>2</sub> in  $\mathcal{T} = (\theta^{>\lambda}, \leq)^I/D$  we have
- (a) every increasing sequence of length  $< \theta$  has an upper bound
  - (b) any increasing sequence of length  $< \theta$  with an upper bound has a lub
  - (c) there is no infinite decreasing sequence so  $(\kappa_1, \kappa_2) \in \mathcal{C}_{\mathcal{T}} \Rightarrow \kappa_2 = 1$ .

[Why? For clause (a) note that  $(\forall x_0, \dots, x_\alpha, \dots)_{\alpha < \delta} (\exists y) (\bigwedge_{\alpha < \beta < \delta} x_\alpha \leq_{\mathcal{T}} x_\beta \rightarrow \bigwedge_{\alpha < \delta} x_\alpha \leq_{\mathcal{T}} y)$  is a Horn sentence. For clause (b) see 2.3, i.e. proof of clause (k) in 2.3.]

- (\*)<sub>3</sub>  $N_1 = N_{\mathbf{r}}^I/D$  is a  $\theta$ -RSP.

[Why? See above recalling 2.2, 2.3.]

- (\*)<sub>4</sub>  $\mathbf{r}$  satisfies  $(k)^+$  from 1.7(3).

[Why? Easily as  $D$  is a  $\aleph_1$ -complete ultrafilter.]

So we are done by 1.7(3). □<sub>2.6</sub>

It is natural to wonder

*Question 2.7.* 1) Is there  $D \in \text{ruf}_{\lambda, \theta}(\lambda)$ , (i.e.  $\theta$ -complete  $(\lambda, \theta)$ -regular ultrafilter on  $\lambda$ ) such that  $\lambda < \mathfrak{t}((\theta^{>\theta}, \leq)^\lambda/D)$ ?

2) Similarly for filters.

3) Use  $\leq_{\mathcal{T}} = \leq$  or  $<_{\mathcal{T}} = \triangleleft$ ?

4) If  $\lambda = \lambda^{<\theta}$ ,  $D$  a fine normal ultrafilter on  $I = [\lambda]^{<\theta}$ , we get  $\lambda \leq \mathfrak{t}(\theta^{>\theta}, \leq)/D$ .

*Remark 2.8.* Now [MiSh:1030, §5] answers 2.7(1) positively for  $\theta$  a super compact cardinal.

**Conclusion 2.9.** Let  $\mathfrak{B}$  be a complete Boolean Algebra and  $D$  a filter on  $\mathfrak{B}$ .

- 1) For every model  $N$ , letting  $\lambda = \|N\| + |\tau_N|$ , we have  $N^{\mathfrak{B}}/D$  is  $(\mu^+, \text{atomically})$ -saturated if  $\mu^+ \leq \min\{p((^{\omega}>\lambda, \leq)^{\mathfrak{B}}/D), t((^{\omega}>\lambda, \leq)^{\mathfrak{B}}/D)\}$ .
- 2) Assume  $\mathfrak{B}$  is  $(\infty, \theta)$ -distributive (e.g. that every decreasing sequence<sup>5</sup> in  $\mathfrak{B}$  of elements from  $Y$  of length  $< \theta$  has a positive lower bound,  $Y \subseteq \mathfrak{B}^+$  is dense) and  $D$  is a  $\theta$ -complete filter on  $\mathfrak{B}$ . If  $N$  is  $(\mu^+, \text{atomically})$ -saturated then  $N^{\mathfrak{B}}/D$  is  $t((^{\omega}>\lambda, \leq)^{\mathfrak{B}}/D)$ -atomically saturated.

*Proof.* As, e.g. in 2.6 above or 2.12 below. □<sub>2.9</sub>

**Conclusion 2.10.** Assume  $(T, \varphi(\bar{x}, \bar{y}))$  has  $\text{SOP}_3$ .

Then, recalling 0.14,  $T$  is  $\leq_{\lambda}^{\text{TP}}$ -maximal for every  $\lambda$  and even  $(T, \{\varphi(\text{bax}, \bar{y})\})$  is.

*Proof.* Should be clear. □<sub>2.10</sub>

\* \* \*

On the connection to Peano arithmetic and to Pabion [Pab82], see Malliaris-Shelah [Sh:F1361]. We repeat some results of [MiSh:1069] in the present context.

**Definition 2.11.** 1) BPA is Bounded Peano Arithmetic.

2)  $N \models \text{BPA}$  is boundedly  $\kappa$ -saturated up to  $(c_1, c_2)$  where  $c_1, c_2, c \in N$  when: if  $p(x) \cup \{x < c_1\}$  is a type in  $N$  (= finitely satisfiable) of cardinality  $< \kappa$  consisting of bounded formulas but with parameters  $\leq c_2$ , then  $p(x) \cup \{x < c_1\}$  is realized in  $N$ .

3) If above  $c_1 = c = c_2$  we may write  $c$  instead of  $(c_1, c_2)$ . We say  $N$  is strongly boundedly  $\kappa$ -saturated up to  $c$  when it holds for  $(c, c_2), c_2 = \infty$ , i.e. we do not bound the parameters.

4) Omitting “up to  $c$ ” in part (3) means for every  $c \in N$ .

**Conclusion 2.12.** Assume  $N$  be a model of BPA.

1) Assume  $a_* \in N$  is non-standard and the power in the  $N$ -sense  $c^{a_*}$  exists for every  $c \in N$ .

For any uncountable cardinal  $\kappa$  the following conditions are equivalent:

- (a)  $N$  is boundedly  $\kappa$ -saturated up to  $c$  for any  $c \in N$
- (b) if  $(C_1, C_2)$  is a cut of  $N$  of cofinality  $(\kappa_1, \kappa_2)$  and  $\kappa_1, \kappa_2$  are infinite (so  $C_1, C_2 \neq 0$ ) then  $\kappa_1 + \kappa_2 \geq \kappa$ .

2) We can weaken the assumption of part (1) by fixing  $c$ , as well as  $N, a_*$ . That is, assume  $N \models “n < a_* \text{ and } c_n = c^{(a_*)^n} \text{ exist}”$  for every standard  $n$  from  $N$ . For every uncountable cardinal  $\kappa$  the following are equivalent:

- (a)'  $N$  is boundedly  $\kappa$ -saturated up to  $c_n$  for each  $n$
- (b)' if  $(C_1, C_2)$  is a cut of  $N$  of cofinality  $(\kappa_1, \kappa_2)$  with  $\kappa_1, \kappa_2$  infinite such that  $c_n \in C_2$  for some  $n$  then  $\kappa_1 + \kappa_2 \geq \kappa$ .

3) Moreover we can add in part (2):

- (c)  $N$  is strongly boundedly  $\kappa$ -saturated up to  $c$ .

---

<sup>5</sup>can weaken the demand



*Proof.* 1) By (2).

2)  $\frac{(a) \Rightarrow (b)}{\text{Trivial.}}$

$\frac{(b) \Rightarrow (a)}$

Without loss of generality  $c$  is not standard (in  $N$ ) and  $n = 0$ . Let  $N^+ = (N, c, a_*)$  and  $\tau^+ = \tau(N^+) = \tau(N) \cup \{c, a_*\}$  and  $\Delta = \{\varphi(x, \bar{y}) \wedge x < c \wedge \bigwedge_{\ell} y_{\ell} < c : \varphi(x, \bar{y}) \in \mathbb{L}(\tau_N) \text{ is a bounded formula}\}$ . We define  $\mathbf{r}$  naturally - the tree of sequences of length  $< a_*$  of members of  $\Delta(N_{\leq c})$  possibly non-standard but of length  $< a_*$ . Now apply 1.7.

3) We just repeat the proof of 1.7 or see 2.16 below.  $\square_{2.12}$

*Question 2.13.* Is  $a_*$  necessary in 2.12(1)? We conjecture that yes.

A partial answer:

**Fact 2.14.** If  $N$  is a model of PA, then  $N$  is  $\kappa$ -saturated iff  $\text{cf}(|N|, <^N) \geq \kappa$  and  $N$  is boundedly  $\kappa$ -saturated.

**Discussion 2.15.** Assume  $N \models \text{BPA}$ ,  $a_1^* <_N B < a_2^*$  and  $B$  is with no last element, we can let

$$A = A_{N, a_*, B} = \{b : \begin{array}{l} b \text{ is } < b' \text{ for some } b' \in B \text{ for some } d, \bar{c} \subseteq B \text{ and } m \text{ and } b \\ \text{belongs to the rang of a function definable in } {}^m(N_{< a_1^*}) \\ \text{definable in } N_{< b} \text{ with parameters from } B \end{array}\}.$$

Now above  $(N \upharpoonright A)_{< b} \prec N_{< b}$  for  $b \in B$ , hence  $N \upharpoonright A \models \text{BPA}$ .

**Claim 2.16.** If (A) then (B) where:

- (A) (a)  $\mathbf{r}_{\alpha}$  is an RSP for  $\alpha < \delta$
- (b)  $\Delta_{\mathbf{r}_{\alpha}} = \Delta$  is a set of quantifier free formulas
- (c)  $\mathcal{T}_{\mathbf{r}_{\alpha}} = \mathcal{T}_{\mathbf{r}_0}$  and  $N_{\mathbf{r}_{\alpha}}$  is increasing with  $\alpha$
- (d)  $Q \in \tau(N_{\mathbf{r}_{\alpha}})$  and  $Q^{N_{\mathbf{r}_{\alpha}}} = Q^{N_{\mathbf{r}_0}}$
- (e) if  $\varphi(x, \bar{y}) \in \Delta_{\mathbf{r}_{\alpha}}$  and  $\bar{b} \in {}^{\ell g(\bar{y})}(N_{\mathbf{r}_{\alpha}})$  then  $\varphi(N_{\mathbf{r}_{\alpha}}, \bar{b}) \subseteq Q^{N_{\mathbf{r}_{\alpha}}}$
- (f)  $\kappa = \min\{\mathbf{p}_{\alpha}(\mathcal{T}_{\mathbf{r}_0}), \mathbf{t}(\mathcal{T}_{\mathbf{r}_0})\}$
- (B) the model  $\cup\{N_{\mathbf{r}_{\alpha}} : \alpha < \delta\}$  is  $(\kappa, 1, \Delta)$ -saturated.

*Proof.* As in 1.7.  $\square_{2.16}$

## § 3. CRITERION FOR ATOMIC SATURATION OF REDUCED POWERS

Malliaris-Shelah [MiSh:998] dealt with such problem for ultrafilters (on sets). The main case here is  $\theta = \aleph_0$ .

**Definition 3.1.** Assume  $D$  is a filter on the complete Boolean Algebra  $\mathfrak{B}$ ,  $T$  an  $\mathbb{L}_{\theta, \theta}(\tau_T)$ -theory,  $\Delta \subseteq \mathbb{L}(\tau_T)$  and  $\mu \geq |\Delta|$ . We say  $D$  is a  $(\mu, \theta, \varepsilon!, \Delta, T)$ -moral filter on  $\mathfrak{B}$  (writing  $\varepsilon$  instead  $\varepsilon!$  means for every  $\varepsilon' < 1 + \varepsilon$ ; if  $\mathfrak{B} = \mathcal{P}(\lambda)$  we may say good instead of moral): when for every  $D - (\mu, \theta, \varepsilon!, \Delta, T)$ -problem there is a  $D - (\mu, \theta, \varepsilon, \Delta, T)$ -solution where:

- (a)  $\bar{\mathbf{a}}$  is a  $D - (\mu, \theta, \varepsilon!, \Delta, T)$ -(moral)-problem when :
  - ( $\alpha$ )  $\bar{\mathbf{a}} = \langle \mathbf{a}_u : u \in [\mu]^{<\theta} \rangle$
  - ( $\beta$ )  $\mathbf{a}_u \in D$  (hence  $\in \mathfrak{B}^+$ )
  - ( $\gamma$ )  $\bar{\mathbf{a}}$  is  $\subseteq$ -decreasing, that is  $u \subseteq v \in [\mu]^{<\theta} \Rightarrow \mathbf{a}_v \leq \mathbf{a}_u$  and  $\mathbf{a}_\emptyset = 1_{\mathfrak{B}}$
  - ( $\delta$ ) for some sequence  $\langle \varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{y}_\alpha) : \alpha < \mu \rangle$  of formulas from  $\Delta$  for every  $\mathbf{a} \in \mathfrak{B} \setminus \{0\}$  and  $u \subseteq \mu$  of cardinality  $< \theta$  we can find  $M \models T$  and  $\bar{b}_\alpha \in {}^{\ell g(\bar{y}_\alpha)} M$  for  $\alpha \in u$  such that:
    - for every  $v \subseteq u$  we have  $\mathbf{a} \leq \mathbf{a}_v \Rightarrow M \models “(\exists \bar{x}_{[\varepsilon]}) \bigwedge_{\alpha \in v} \varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{b}_\alpha)”$   
and  $\mathbf{a} \leq 1 - \mathbf{a}_v \Rightarrow M \models “\neg(\exists \bar{x}_{[\varepsilon]}) \bigwedge_{\alpha \in v} \varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{b}_\alpha)”$
- (b)  $\bar{\mathbf{b}}$  is a  $D - (\mu, \theta)$ -(moral)-solution of the  $D - (\mu, \theta, \varepsilon, \Delta, T)$ -(moral)-problem  $\bar{\mathbf{a}}$  when
  - ( $\alpha$ )  $\bar{\mathbf{b}} = \langle \mathbf{b}_u : u \in [\mu]^{<\theta} \rangle$
  - ( $\beta$ )  $\mathbf{b}_u \in D$  and  $\mathbf{b}_\emptyset = 1_{\mathfrak{B}}$
  - ( $\gamma$ )  $\mathbf{b}_u \leq \mathbf{a}_u$
  - ( $\delta$ )  $\bar{\mathbf{b}}$  is multiplicative, i.e.  $\bar{\mathbf{b}}_u = \cap \{\mathbf{b}_{\{\alpha\}} : \alpha \in u\}$ .

*Remark 3.2.* 1) The  $\theta$  here means “a type is  $(< \theta)$ -satisfiable”.

2) The use of “ $\varepsilon!$ ” is to conform with Definition 0.10.

Recall (from 0.10)

**Definition 3.3.** Let  $\tau$  be a vocabulary and  $\Delta \subseteq \{\varphi \in \mathbb{L}(\tau) : \varphi = \varphi(\bar{x}, \bar{y})\}$  but  $\varphi(\bar{x}, \bar{y}) \in \Delta$  means we can add to  $\bar{x}$  dummy variables. Let  $\lambda > \theta$  (dull otherwise).

A  $\tau$ -model  $M$  is  $\langle \lambda, \theta, \varepsilon!, \Delta \rangle$ -saturated when: if  $p \subseteq \{\varphi(\bar{x}_{[\varepsilon]}, \bar{a}) : \varphi(\bar{x}_{[\varepsilon]}, \bar{y}) \in \Delta, \bar{a} \in {}^{\ell g(\bar{y})} M\}$  has cardinality  $< \lambda$  is  $(< \theta)$  satisfiable in  $M$  then  $p$  is realized in  $M$ .

**Claim 3.4.** 1) For a  $(\mu, \theta)$ -regular  $\theta$ -complete ultra-filter  $D$  on a set  $I$  and  $\theta$ -saturated or just  $(\theta, \aleph_0, \varepsilon!, \Delta)$ -saturated model  $M$ , a cardinality  $\mu$  and  $\Delta = \mathbb{L}_{\theta, \theta}(\tau_M)$ , the following conditions are equivalent:

- (a)  $D$  is  $(\mu, \theta, \varepsilon!, \Delta, T)$ -moral ultrafilter on the Boolean Algebra  $\mathcal{P}(T)$
- (b) if  $M \in \text{Mod}_T$  then  $M^I/D$  is  $(\mu, \theta, \varepsilon!, \Delta, T)$ -saturated.

2) Similarly for  $D$  a filter on a  $\theta$ -distributive (see 0.11(8)) complete Boolean Algebra  $\mathfrak{B}$ .

*Proof.* Similar to 3.5 it actually follows from it because as  $D$  is an ultra-filter, we can start with  $M \models T$ , expand it to  $M^+$  by adding a predicate to any definable relation and apply 3.5 to  $T^+ = \text{Th}(M^+)$ .  $\square_{3.4}$

**Claim 3.5.** 1) If (A) then (B)  $\Leftrightarrow$  (C) where:

- (A) (a)  $\mathfrak{B} = \mathcal{P}(I)$
- (b)  $D$  is a  $\theta$ -complete  $(\mu, \theta)$ -regular filter on  $\mathfrak{B}$
- (c)  $\mu = \mu^{<\theta} > \varepsilon$  or  $\theta > \varepsilon$
- (d)  $T$  is an  $\mathbb{L}_{\theta, \theta}(\tau)$ -theory
- (e)  $\Delta$  is a set of conjunctions of  $< \theta$  atomic formulas from  $\mathbb{L}_{\theta, \theta}(\tau)$
- (B)  $D$  is a  $(\mu, \theta, \varepsilon!, \Delta, T)$ -moral filter on  $\mathfrak{B}$
- (C) if  $M_s$  is a model of  $T$  for  $s \in I$  then  $\prod_{s \in I} M_s / D$  is  $(\mu^+, \theta, \varepsilon!, \Delta, T)$ -saturated.

2) If (A)' then (B)'  $\Leftrightarrow$  (C)' where

- (A)' (a)  $\mathfrak{B}$  is a  $\theta$ -distributive (see 0.11(8)) complete Boolean Algebra
- (b) – (e) as above (on regularity see Definition 0.12) but sn
- (d)<sup>+</sup>  $T$  is a complete  $\mathbb{L}_{\theta, \theta}(\tau)$ -theory
- (B)' as (B) above
- (C)' (a) if  $M$  is a model of  $T$  then  $M^{\mathfrak{B}} / D$  is  $(\mu^+, \theta, \varepsilon!, \Delta)$ -saturated
- (b) if  $\mathcal{J}$  is a maximal antichain of  $\mathfrak{B}$  and  $\bar{M} = \langle M_b : b \in \mathcal{J} \rangle$  is a sequence of  $\tau$ -models then  $\bar{M}^{\mathfrak{B}} / D$  is  $(\mu^+, \theta, \varepsilon!, \Delta)$ -saturated.

*Proof.* 1) Proving (B)  $\Rightarrow$  (C): Let  $N = \prod_{s \in I} M_s / D$  let  $\bar{x} = \bar{x}_{[\varepsilon]}$ ,  $\varphi_\alpha = \varphi_\alpha(\bar{x}, \bar{y}_\alpha)$  and assume that  $p(\bar{x}) = \{\varphi_\alpha(\bar{x}, \bar{b}_\alpha) : \alpha < \alpha_*\}$  is  $(< \theta)$ -satisfiable in  $N$  and  $|\alpha_*| \leq \mu$ , so without loss of generality  $\alpha_* = \mu$ ; without loss of generality let  $\varphi_\alpha = \varphi_\alpha(\bar{x}, \bar{y}_{[\xi_\alpha]})$  so  $\bar{b}_\alpha \in \xi_\alpha(\prod_{s \in I} M_s)$ .

Let  $\bar{b}_\alpha = \langle f_{\alpha, \xi} / D : \xi < \xi_\alpha \rangle$  where  $f_{\alpha, \xi} \in \prod_{s \in I} M_s$  and for  $s \in I$  let  $\bar{b}_{\alpha, s} = \langle f_{\alpha, \xi}(s) : \xi < \xi_\alpha \rangle$ ; now for  $u \in [\mu]^{<\theta}$  we let

$$(*)_0 \mathbf{a}_u := \{s \in I : M_s \models (\exists \bar{x}) \bigwedge_{\alpha \in u} \varphi_\alpha(\bar{x}, \bar{b}_{\alpha, s})\}.$$

Now

$$(*)_1 \bar{\mathbf{a}} = \langle \mathbf{a}_u : u \in [\mu]^{<\theta} \rangle \text{ is a } D - (\mu, \theta, \varepsilon!, \Delta, T)\text{-problem.}$$

[Why? We should check Definition 3.1, clause (a): now (a)( $\alpha$ ) is trivial;  $\mathbf{a}_u \subseteq I$  holds by the choice of  $\mathbf{a}_\alpha$ . Toward clause (a), ( $\beta$ ) fix a set  $u \in [\mu]^{<\theta}$ ; some  $\bar{c} \in {}^\varepsilon N$  realizes the type  $p_u(\bar{x}_{[\varepsilon]}) = \{\varphi_\alpha(\bar{x}, \bar{b}_\alpha) : \alpha \in u\}$  in  $N$ , see Definition 3.3, so let  $\bar{c} = \langle g_\zeta / D : \zeta < \varepsilon \rangle$  for some  $g_\zeta \in \prod_{s \in I} M_s$  for  $\zeta < \varepsilon$  and let  $\bar{c}_s = \langle g_\zeta(s) : \zeta < \varepsilon \rangle \in {}^\varepsilon(M_s)$ . So  $\mathbf{a}'_{\{\alpha\}} = \{s \in I : M \models \varphi_\alpha[\bar{c}_s, \bar{b}_s]\}$  belong to  $D$  because  $N \models \varphi_\alpha[\bar{c}, \bar{b}_\alpha]$  by the definition of  $N$  if  $\varphi_\alpha$  is atomic, but recalling  $D$  is  $\theta$ -complete also for our  $\varphi_\alpha$ , remembering clause (A)(a) of 3.5(1). As  $D$  is  $\theta$ -complete clearly,  $\mathbf{a}'_u = \cap \{\mathbf{a}'_{\{\alpha\}} : \alpha \in u\}$  belongs to  $D$  and by our choices,  $\mathbf{a}'_u \leq_{\mathfrak{B}} \mathbf{a}_u$ , hence  $\mathbf{a}_u \in D$  so subclause (a)( $\beta$ ) holds indeed.

By the choice of  $\mathbf{a}_u$ ,  $\mathbf{a}_u$  is  $\subseteq$ -decreasing with  $u$  so subclause (a)( $\gamma$ ) holds.

Lastly, subclause (a)( $\delta$ ) holds by the definition of  $\mathbf{a}_u$ 's recalling  $p(\bar{x})$  is ( $< \theta$ )-satisfiable.]

(\*)<sub>2</sub> there is  $\bar{\mathbf{b}}$ , a  $D - (\mu, \theta)$ -solution of  $\mathbf{a}$  in  $\mathfrak{B}$ .

[Why? Because we are presently assuming clause (B) of 3.5 which says that  $D$  is  $(\mu, \theta, \varepsilon!, \Delta, T)$ -good, see Definition 3.1.]

(\*)<sub>3</sub> without loss of generality  $s \in I \Rightarrow \{\alpha < \mu : s \in \mathbf{b}_{\{\alpha\}}\}$  has cardinality  $< \theta$ .

[Why? As  $D$  is  $(\mu, \theta)$ -regular.]

Next for  $s \in I$  let  $u_s = \{\alpha < \mu : s \in \mathbf{b}_{\{\alpha\}}\}$  but  $\bar{\mathbf{b}}$  is multiplicative (see 3.1(b)( $\delta$ )) so  $\mathbf{b}_{u_s} = \cap \{\mathbf{b}_{\{\alpha\}} : \alpha \in u_s\} = \cap \{\mathbf{b}_{\alpha} : \text{the ordinal } \alpha \text{ satisfies } s \in \mathbf{b}_{\{\alpha\}}\}$  hence  $s \in \mathbf{b}_{u_s}$  hence (see 3.1(b) recalling that  $|u_s| < \theta$  by (\*)<sub>2</sub>) we have  $s \in \mathbf{a}_{u_s}$  hence (by the choice of  $\mathbf{a}_{u_s}$ ) there is  $\bar{a}_s \in {}^\varepsilon(M_s)$  realizing  $\{\varphi(\bar{x}_{[\varepsilon]}, \langle f_{\alpha, \varepsilon_\zeta}(s) : \zeta < \varepsilon \rangle) : \alpha \in u_s\}$ .

Let  $\bar{a}_s = \langle a_{s, \zeta} : \zeta < \varepsilon \rangle$ . Now for  $\zeta < \varepsilon = \ell g(\bar{x})$  let  $g_\zeta \in \prod_{s \in I} M_s$  be defined by  $g_\zeta(s) = a_{s, \zeta} \in M_s$  and let  $\bar{a} = \langle g_\zeta / D : \zeta < \varepsilon \rangle$  noting  $g_\zeta / D \in \prod_{s \in I} M_s / D = N$ . Hence for every  $\alpha < \mu$ ,  $\{s \in I : M_s \models \varphi_\alpha(\langle g_\zeta(s) : \zeta < \varepsilon \rangle, \bar{b}_{\alpha, s})\} \supseteq \mathbf{b}_{\{\alpha\}} \in D$  so  $N \models \varphi[\bar{a}, \bar{b}_\alpha]$ .

Hence  $\bar{a}$  realizes  $p(\bar{x})$  in  $N$  as promised.

Proving (C)  $\Rightarrow$  (B):

So let  $\bar{\mathbf{a}}$  be a  $D - (\mu, \theta, \varepsilon!, \Delta, T)$ -problem and let  $\bar{\varphi} = \langle \varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{y}_\alpha) : \alpha < \mu \rangle$  be a sequence of formulas from  $\Delta$  as in clause (a)( $\delta$ ) of Definition 3.1.

Let  $\bar{w} = \langle w_s : s \in I \rangle$  be a sequence of subsets of  $\mu$  each of cardinality  $< \theta$  such that  $\alpha < \mu \Rightarrow \{s \in I : \alpha \in w_s\} \in D$ . For  $u \in [\mu]^{<\theta}$  let  $\mathbf{c}_u = \{s \in I : u \subseteq w_s\}$ , so clearly  $\mathbf{c}_u \in D$  and  $\langle \mathbf{c}_u : u \in [\mu]^{<\theta} \rangle$  is multiplicative.

For each  $s \in I$  applying Definition 3.1(a)( $\delta$ ) to  $\mathbf{a} = \{s\}$  and  $u = w_s$  we can find a model  $M_s$  of  $T$  and  $\bar{b}_{s, \alpha} \in {}^{\ell g(\bar{y}_\alpha)}(M_s)$  for  $\alpha \in w_s$  satisfying  $\bullet$  there.

Now choose  $\bar{b}_{s, \alpha}$  also for  $s \in I, \alpha \in \mu \setminus w_s$ , as any sequence of members of  $M_s$  of length  $\ell g(\bar{y}_\alpha)$ . Now for every  $\alpha < \mu$  and  $j < \ell g(\bar{y}_\alpha)$  we define  $g_{\alpha, j} \in \prod_{s \in I} M_s$  by

$$g_{\alpha, j}(s) = (\bar{b}_{s, \alpha})_j.$$

Hence  $g_{\alpha, \zeta} / D \in \prod_{s \in I} M_s / D = N$  and  $\bar{b}_\alpha = \langle g_{\alpha, \zeta} / D : \zeta < \ell g(\bar{y}_\alpha) \rangle \in {}^{\ell g(\bar{y}_\alpha)}N$  and consider the set  $p = \{\varphi_\alpha(\bar{x}, \bar{b}_\alpha) : \alpha < \mu\}$ . Is  $p$  a ( $< \theta$ )-satisfiable type in  $N$ ? Yes, because if  $u \in [\mu]^{<\theta}$ , then  $\mathbf{c}_u \in D$  and  $s \in \mathbf{c}_u \Rightarrow \{\varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{b}_{s, \alpha}) : \alpha \in u\}$  is realized in  $M_s$ , say by  $\bar{a}_s = \langle a_{s, \zeta} : \zeta < \varepsilon \rangle$ ; for  $s \in I \setminus \mathbf{c}_u, \zeta < \varepsilon$  let  $a_{s, j} \in M$  be arbitrary and let  $f_{s, \zeta} \in \prod_{s \in I} M_s$  be  $f_{s, \zeta}(s) = a_{s, j}$ . Easily  $\langle f_{\alpha, j} / D : \zeta < \varepsilon \rangle$  realizes  $p$  because  $\mathbf{a}_u \cap \mathbf{c}_u \in D$ .

Next, we apply clause (c) we are assuming so  $p(\bar{x}_{[\varepsilon]})$  is realized in  $N$ . So let  $\bar{a} = \langle a_\zeta : \zeta < \varepsilon \rangle \in {}^\varepsilon N$  realize  $p$  and let  $a_\zeta \in h_\zeta / D$  where  $h_\zeta \in \prod_{s \in I} M_s$  and lastly let

$$\mathbf{b}_u = \{s \in I : M_s \models \varphi_\alpha(\langle h_\zeta(s) : \zeta < \varepsilon \rangle, \bar{b}_{s, \alpha}) \text{ for every } \alpha \in u \text{ and } s \in \mathbf{c}_u\}.$$

Now check that  $\langle \mathbf{b}_u : u \in [\mu]^{<\theta} \rangle$  is as required, recalling  $\langle \mathbf{c}_u : u \in [\mu]^{<\theta} \rangle$  is multiplicative. So the desired conclusion of 3.1(B) holds indeed so we are done proving (C)  $\Rightarrow$  (B).

2) Similarly; e.g. for clause (a) let  $p(\bar{x})$  be as there but

- $f_{\alpha,\xi} \in M^{\mathfrak{B}}$  is supported by the maximal antichain  $\langle c_{\alpha,\xi,i} : i < i(\alpha,\xi) \rangle$
- $(*)_0$   $\mathbf{a}_u = \sup\{\mathbf{c} : \text{we have } \alpha \in u \wedge \xi < \xi_\alpha \Rightarrow (\exists \mathbf{d})(\mathbf{d} \in \text{Dom}(f_{\alpha,\xi}) \wedge \mathbf{c} \leq \mathbf{d}) \text{ and } M \models (\exists \bar{x}_{[\varepsilon]}) \bigwedge_{\alpha \in u} \varphi(\bar{x}_{[\varepsilon]}, \langle f_{\alpha,\xi}(\mathbf{c}) : \xi < \xi_\alpha \rangle)\}$
- $(*)_1$   $\bar{\mathbf{a}} = \langle \mathbf{a}_u : u \in [\mu]^{<\theta} \rangle$  is a  $D - (\mu, \theta, \varepsilon!, \Delta, T)$ -problem.

[Why? As there.]

- $(*)_2$  let  $\bar{\mathbf{b}}$  be a  $D - (\mu, \theta, \varepsilon!, \Delta, T)$ -solution.

[Why exists? By  $(B)'$  recalling Definition 3.1.]

Also the rest is as above. □<sub>3.5</sub>

*Remark 3.6.* If  $\mathcal{S} \subseteq [\mu]^{<\theta}$  is cofinal,  $u \in [\mu]^{<\theta} \Rightarrow |\mathcal{P}(u)| \cap \mathcal{S}| < \theta_1$  we may consistently replace  $[\mu]^{<\theta}$  by  $\mathcal{S}$  and  $2^{\theta_1}$  by  $\theta_1$ .

**Definition 3.7.** 1) A filter  $D$  on a complete Boolean Algebra  $\mathfrak{B}$  is  $(\mu, \theta)$ -excellent when: if  $\bar{\mathbf{a}} = \langle \mathbf{a}_u : u \in [\mu]^{<\theta} \rangle$  is a sequence of members of  $\mathfrak{B}$ , (yes! not necessarily from  $D$ ) then we can find  $\bar{\mathbf{b}}$  which is a multiplicative refinement of  $\bar{\mathbf{a}}$  for  $D$ , meaning:

- (a)  $\bar{\mathbf{b}} = \langle \mathbf{b}_u : u \in [\mu]^{<\theta} \rangle$
- (b)  $\mathbf{b}_u \leq \mathbf{a}_u$  and  $\mathbf{b}_u = \mathbf{a}_u \pmod D$
- (c) if  $\mathbf{a}_{u_1} \cap \mathbf{a}_{u_2} = \mathbf{a}_{u_1 \cap u_2} \pmod D$  then  $\mathbf{b}_{u_1} \cap \mathbf{b}_{u_2} = \mathbf{b}_{u_1 \cap u_2}$ .

2) For a Boolean algebra  $\mathfrak{B}$  and filter  $D$  on  $\mathfrak{B}$  we say  $\bar{\mathbf{a}}$  is a  $D - (\mu, \theta)$ -problem when clauses (a)( $\alpha$ ), ( $\beta$ ), ( $\gamma$ ) of Definition 3.1 holds.

**Claim 3.8.** 1) The filter  $D$  on  $I$  (i.e. on the Boolean Algebra  $\mathcal{P}(I)$ ) is  $(\mu, \theta, \varepsilon!, \Delta, T)$ -moral if the filter  $D_1$  on the complete Boolean Algebra  $\mathfrak{B}_1$  is  $(\mu, \theta, \varepsilon!, \Delta, T)$ -moral when:

- (a)  $\mathbf{j}$  is a homomorphism from  $\mathcal{P}(I)$  onto  $\mathfrak{B}_1$
- (b)  $D_0 = \{A \subseteq I : \mathbf{j}(A) = 1_{\mathfrak{B}_1}\}$  is a  $(\mu, \theta)$ -excellent filter on  $I$ , moreover is  $(2^\theta)^+$ -complete for every  $\partial < \theta$
- (c)  $D = \{A \subseteq I : \mathbf{j}(A) \in D_1\}$ .

2) We can replace  $\mathcal{P}(I)$  by a complete Boolean Algebra  $\mathfrak{B}_2$ .

*Proof.* The “if” direction:

We assume  $D_1$  is  $(\mu, \theta, \varepsilon!, \Delta, T)$ -moral and should prove it for  $D$ . So let  $\bar{A} = \langle A_u : u \in [\mu]^{<\theta} \rangle$  be a  $D - (\mu, \theta, \varepsilon!, \Delta, T)$ -problem and we should find a  $D - (\mu, \theta)$ -solution  $\bar{B}$  of it.

Clearly  $\mathbf{a}_u := \mathbf{j}(A_u) \in \mathfrak{B}^+$  and  $\bar{\mathbf{a}} = \langle \mathbf{a}_u : u \in [\mu]^{<\theta} \rangle = \langle \mathbf{j}(A_u) : u \in [\mu]^{<\theta} \rangle$  is a  $D_1 - (\mu, \theta, \varepsilon, \Delta, T)$ -problem.

Hence by our present assumption ( $D_1$  is  $(\mu, \theta, \varepsilon!, \Delta, T)$ -moral) there is a  $D_1 - (\mu, \theta)$ -solution  $\bar{\mathbf{b}}$  of  $\bar{\mathbf{a}}$ , let  $\bar{\mathbf{b}} = \langle \mathbf{b}_u : u \in [\mu]^{<\theta} \rangle$  so  $u \in [\mu]^{<\theta} \Rightarrow \mathbf{b}_u \in D_1$ . For  $u \in [\mu]^{<\theta}$  choose  $B_u^1 \subseteq A_u$  be such that  $\mathbf{j}(B_u^1) = \mathbf{b}_u$ , possible because  $\mathbf{j}$  is a homomorphism from  $\mathcal{P}(I)$  onto  $\mathfrak{B}_1$ . So  $\bar{B}^1 = \langle B_u^1 : u \in [\mu]^{<\theta} \rangle$  is a multiplicative modulo  $D_0$ , i.e.  $\langle B_u^1/D_0 : u \in [\mu]^{<\theta} \rangle$  is a multiplicative sequence of members of  $\mathcal{P}(I)/D_0$ .

Let  $B_n^2 = B_u^1 \cap A_u$ , so

- $B_u^1 \subseteq A_u \pmod{D_0}$ .

[Why? As  $\mathbf{j}(B_u^1) = \mathbf{b}_u \leq \mathbf{a}_u = \mathbf{j}(A_u)$ .]

- $B_u^2 \subseteq B_u^1$  and  $B_u^2 \subseteq A_u \pmod{D_0}$
- $B_u^2 \in D$
- $\langle B_u^2 : u \in [\mu]^{<\theta} \rangle$  is multiplicative for  $D_0$  (see 3.7).

By Definition 3.7 applied to  $\langle B_u^2 : u \in [\mu]^{<\theta} \rangle$  recalling clause (b) of the assumption of the claim, we can find  $\bar{B} = \langle B_u : u \in [\mu]^{<\theta} \rangle$  which is a multiplicative refinement of  $\bar{B}^2$  for  $D_0$  and is multiplicative.

So we are done for the “if” direction.

The “only if” direction:

So we are assuming  $D$  is a  $(\mu, \theta, \varepsilon, \Delta, T)$ -good filter on  $\lambda$  and we have to prove  $D_1$  is  $(\mu, \theta, \varepsilon, \Delta, T)$ -moral.

So let  $\bar{\mathbf{a}}$  be a  $D_1 - (\mu, \theta, \varepsilon, \Delta, T)$ -moral problem (on  $\mathfrak{B}_1$ ), we have to find a solution. For  $u \in [\mu]^{<\theta}$  choose  $A_u^1 \subseteq \lambda$  such that  $\mathbf{j}(A_u^1) = \mathbf{a}_u$ ; by 3.7, i.e. clause (b) of the assumption of the claim there is  $\bar{A}^2 = \langle A_u^2 : u \in [\mu]^{<\theta} \rangle$  such that  $A_u^2 \subseteq A_u^1, A_u^2 = A_u^1 \pmod{D_0}$  hence  $A_u^2 \in D$  and  $A_u^1$  is  $\subseteq$ -decreasing with modulo  $D_0$ . [Why? Because  $\bar{A}$  is  $\subseteq$ -decreasing as  $\bar{\mathbf{a}}$  is  $\leq$ -decreasing hence  $\bar{A}^2$  is  $\subseteq$ -decreasing modulo  $D_0$ .]

As  $D$  is  $(\mu, \theta, \varepsilon, \Delta, T)$ -good filter on  $I$  there is a  $D$ -multiplicative refinement  $\langle B_u^2 : u \in [u]^{<\theta} \rangle$ . Let  $\mathbf{b}_u = \mathbf{j}(B_u^2)$ , now  $\langle \mathbf{b}_u : u \in [\mu]^{<\theta} \rangle$  is as required.

2) Similarly.

□3.8

**Claim 3.9.** 1)  $D$  is  $(\mu, \theta)$ -excellent implies  $D$  is  $(\mu, \theta)$ -good.

2)  $D$  is  $(\mu, \theta)$ -good implies  $D$  is  $(\mu, \theta, \varepsilon, \Delta, T)$ -moral.

*Proof.* 1) So let  $\bar{\mathbf{a}} = \langle \mathbf{a}_u : u \in [\mu]^{<\theta} \rangle$  be a  $D$ -problem and we should find a  $D - (\mu, \theta)$ -solution  $\bar{\mathbf{b}}$  below  $\bar{\mathbf{a}}$ . As  $D$  is  $(\mu, \theta)$ -excellent we apply this to  $\bar{\mathbf{a}}$  and  $\bar{\mathbf{b}}$  as in 3.7(2). Easily it is as required.

2) Just read the definitions: there are fewer problems.

□3.9

*Remark 3.10.* We may wonder, e.g. in 3.5(1), can we move the regularity demand on the filter  $D$  from clause (A) to clause (B) The answer is yes for most  $T$ 's.

**Claim 3.11.** The filter  $D$  is  $(\lambda, \theta)$ -regular when:

- (A) (a)  $\mathfrak{B} = \mathcal{P}(I)$
- (b)  $D$  is a  $\sigma$ -complete ultrafilter on  $\mathfrak{B}$
- (c)  $\mu = \mu^{<\theta} > \varepsilon$  or  $\theta > \varepsilon$  (check!)
- (d)  $T$  is a complete  $\mathbb{L}_{\theta, \theta}(\tau)$ -theory, e.g.  $T = \text{Th}_{\mathbb{L}_{\theta, \theta}}(M)$ ,  $M$  a  $\theta$ -saturated model (note that  $T = T_0^{[\theta]}$  where  $T_0 = \text{Th}_{\mathbb{L}_{\aleph_0, \aleph_0}}(M)$ , i.e.  $T$  is determined by  $T_0$  and  $\theta$ )
- (B)  $T$  has a model  $M$  and  $p = \{\varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{b}_\alpha) : \alpha < \mu\}, \varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{y}_\alpha) \in \mathbb{L}_{\theta, \theta}, \bar{b}_\alpha \in {}^{\ell g(\bar{y}_\alpha)} M$  such that: for every  $q \subseteq p$ 
  - $q$  is realized in  $M$  iff  $|q| < \theta$
- (C) if  $M_s$  is a model of  $T$  for  $s \in I$  then  $\prod_s M_s / D$  is  $(\mu^+, \theta, \varepsilon!, \Delta, T)$ .

*Proof.* Should be clear.

□3.11

## § 4. COUNTEREXAMPLE

 § 4(A). On The parallel of  $\mathfrak{p} = \mathfrak{t}$ .

In §2 we generalize [Sh:c, Ch.VI,2.6] to filters, using the class of relevant RSP's  $\mathbf{r}$  being closed under reduced powers (being a Horn class, see 2.2). Can we generalize the result of Malliaris-Shelah [MiSh:998]? It seems that we can give a counterexample.

For this we have to find

- (\*)<sub>1</sub>  $D$  a filter of  $\lambda$  such that the partial order  $N_1 = (\mathbb{Q}, <)^\lambda / D$  satisfies  $\mathfrak{p}^*(N_1) = \kappa_1 + \kappa_2 < \mu^+ \leq \mathfrak{p}_{\text{sym}}^*(N_1)$ ,  $\kappa_1 \neq \kappa_2$ ,  $(\kappa_1, \kappa_2) \in \mathcal{C}(N_1)$ , so in fact  $N_1$  have no  $(\theta_1, \theta_2)$ -cut when  $\theta_1 = \text{cf}(\theta_1) = \theta_2 \leq \mu$  and when  $\theta_\ell \geq \mu^+ \wedge \theta_{3-\ell} \in \{0, 1\}$
- (\*)<sub>2</sub> preferably:  $\lambda = \mu$
- (\*)<sub>3</sub> or at least for some dense linear order  $M_0$  there is a complete Boolean Algebra  $\mathfrak{B}$  and a filter  $D$  on  $\mathfrak{B}$  such that  $N_0 = M_0^\mathfrak{B} / D$  is as above.

We presently deal with the (main) case  $\theta = \aleph_0$  and carry this out. It seems reasonable that we can prove, e.g.  $T_{\text{feq}} \not\leq_{\text{rp}} T_{\text{ord}}$  but we have not arrived to it. In ?? we say more; we can control the set of non-symmetric pre-cuts.

**Convention 4.1.**  $T_{\text{ord}}$  is the first order theory of  $(\mathbb{Q}, \leq)$ , see 4.3(1)(d).

**Definition 4.2.** Let  $\kappa$  be a regular cardinal.

1) Let  $K_\kappa^{\text{ba}}$  be the class of  $\mathbf{m}$  such that:

- (a)  $\mathbf{m} = (\mathfrak{B}, D) = (\mathfrak{B}_\mathbf{m}, D_\mathbf{m})$
- (b)  $\mathfrak{B}$  is a complete Boolean Algebra satisfying the  $\kappa$ -c.c.
- (c)  $D$  is a filter on  $\mathfrak{B}$ .

2) Let  $\leq_\kappa^{\text{ba}}$  be the following two-place relation on  $K_\kappa^{\text{ba}}$ :  $\mathbf{m} \leq_\kappa^{\text{ba}} \mathbf{n}$  iff

- (a)  $\mathbf{m}, \mathbf{n} \in K_\kappa^{\text{ba}}$
- (b)  $\mathfrak{B}_\mathbf{m} \leq \mathfrak{B}_\mathbf{n}$
- (c)  $D_\mathbf{m} = D_\mathbf{n} \cap \mathfrak{B}_\mathbf{m}$ .

3) Let  $S_\kappa^{\text{ba}}$  be the class of  $\leq_\kappa^{\text{ba}}$ -increasing continuous sequences  $\bar{\mathbf{m}}$  which means:

- (a)  $\bar{\mathbf{m}} = \langle \mathbf{m}_\alpha : \alpha < \ell g(\bar{\mathbf{m}}) \rangle$
- (b)  $\mathbf{m}_\alpha \in K_\kappa^{\text{ba}}$
- (c) if  $\alpha < \beta < \ell g(\bar{\mathbf{m}})$  then  $\mathbf{m}_\alpha \leq_\kappa^{\text{ba}} \mathbf{m}_\beta$
- (d) if  $\beta < \ell g(\bar{\mathbf{m}})$  is a limit ordinal then:
  - (α)  $\mathfrak{B}_{\mathbf{m}_\beta}$  is the completion of  $\cup \{ \mathfrak{B}_{\mathbf{m}_\alpha} : \alpha < \beta \}$
  - (β)  $D_{\mathbf{m}_\beta}$  is generated (as a filter) by  $\cup \{ D_{\mathbf{m}_\alpha} : \alpha < \beta \}$ .

4) If  $\kappa = \aleph_1$  we may write  $K_{\text{ba}}, \leq_{\text{ba}}, S_{\text{ba}}$ .

**Definition 4.3.** Let  $\mathbf{m} \in K_{\text{ba}}$ .

1) We say  $\bar{\mathbf{a}}$  is a  $T_{\text{ord}} - (\kappa_1, \kappa_2)$ -moral problem in  $\mathbf{m}$  when:

- (a)  $\mathbf{m} \in K_\kappa^{\text{ba}}$
- (b)  $I = I(\kappa_1, \kappa_2)$  is the linear order  $I_1 + I_2$  where

- $I_1 = I_1(\kappa_1) = (\{1\} \times \kappa_1)$ ,
  - $I_2 = I_2(\kappa_2) = (\{2\} \times \kappa_2^*)$
- (c)  $\bar{\mathbf{a}} = \langle \mathbf{a}_{s,t} : s <_{I(\kappa_1, \kappa_2)} t \rangle$  is a sequence of members of  $D_{\mathbf{m}}$
- (d) if  $u \subseteq I$  is finite,  $\mathbf{t} : u \times u \rightarrow \{0, 1\}$  and  $\cap \{\mathbf{a}_{s,t}^{\text{if}(\mathbf{t}(s,t))} : s, t \in u\} > 0$  then there is a function  $f : u \rightarrow \{0, \dots, |u| - 1\}$  such that:
- if  $s, t \in u$  then  $\mathbf{t}(s, t) = 1$  iff  $f(s) \leq f(t)$
- (e) hence  $s_1 <_I s_2 <_I s_2 \Rightarrow \mathbf{a}_{s_1, s_2} \cap \mathbf{a}_{s_2, s_3} \leq \mathbf{a}_{s_1, s_3}$  and we stipulate  $\mathbf{a}_{s,s} = 1_{\mathfrak{B}}, \mathbf{a}_{t,s} = \mathbf{a}_{s,t}$  when  $s <_I t$ .
- 2) We say  $\bar{\mathbf{b}}$  is a solution of  $\bar{\mathbf{a}}$  in  $\mathbf{m}$  where  $\bar{\mathbf{a}}$  is as above when:
- (a)  $\bar{\mathbf{b}} = \langle \mathbf{b}_s : s \in I \rangle$
  - (b)  $\mathbf{b}_s \in D_{\mathbf{m}}$
  - (c) if  $s_1 \in I_1, s_2 \in I_2$  then  $\mathbf{b}_{s_1} \cap \mathbf{b}_{s_2} \leq \mathbf{a}_{s_1, s_2}$ .
- 3)  $\mathbf{S}$  is the class of tuples  $\mathbf{s} = (I, D_0, \mathbf{j}, \mathfrak{B}, D_1, D)$  such that
- (a)  $\mathbf{j}$  is a homomorphism from  $\mathcal{P}(I)$  onto the complete Boolean  $\mathfrak{B}$
  - (b)  $D_1$  is a filter on  $\mathfrak{B}$
  - (c)  $D_0 = \{A \subseteq I : \mathbf{j}(A) = 1_{\mathfrak{B}}\}$  (or see §3)
  - (d)  $D = \{A \subseteq I : \mathbf{j}(A) \in D_1\}$ .
- 3A) For  $\mathbf{s} \in \mathcal{S}$  let  $\mathbf{m}_{\mathbf{s}} = (\mathfrak{B}_{\mathbf{s}}, D_1)$ .
- 4) We say  $\mathbf{s} \in \mathbf{S}$  is  $(\mu, \theta)$ -excellent (if  $\theta = \aleph_0$  may omit) when  $D_0$  is an excellent filter on  $I$ , see Definition 3.7(2).
- 5) We say  $\mathbf{s} \in \mathbf{S}$  is  $(\mu, \theta)$ -regular (if  $\theta = \aleph_0$  we may omit  $\theta$ ) when  $D_0$  is a  $(\mu, \theta)$ -regular filter.
- 6) Let  $\mathbf{S}_{\mu, \theta}$  be the class of  $(\mu, \theta)$ -excellent  $(\mu, \theta)$ -regular  $\mathbf{s} \in \mathbf{S}$ ; we may omit  $\theta$  if  $\theta = \aleph_0$ .
- 7) Convention: if  $\bar{\mathbf{a}}$  as above is given, let  $I_1, I_2$  be as above.

**Claim 4.4.** 1) For  $\mathbf{m} = (\mathfrak{B}, D) \in K_{\text{ba}}$  and  $\kappa_1, \kappa_2$  are infinite and regular cardinals we have: for some  $M \in \text{Mod}_{T_{\text{ord}}}, M^{\mathfrak{B}}/D$  has a  $(\kappa_1, \kappa_2)$ -pre-cut iff some  $T_{\text{ord}} - (\kappa_1, \kappa_2)$ -moral problem in  $\mathbf{m}$  has no solution.

2) If  $\mathbf{s} \in \mathbf{S}_{\mu, \theta}$  so is  $\mu$ -excellent and  $\mu$ -regular and  $\kappa_1, \kappa_2 \geq \aleph_0$  are regular and  $\kappa_1 + \kappa_2 \leq \mu$  then the following conditions are equivalent:

- (a) for some linear order  $M, M^{I(\mathbf{s})}/D_{\mathbf{s}}$  has a  $(\kappa_1, \kappa_2)$ -pre-cut
- (b) for every infinite linear order,  $M^{I(\mathbf{s})}/D_{\mathbf{s}}$  has a  $(\kappa_1, \kappa_2)$ -pre-cut
- (c) every  $T_{\text{ord}} - (\kappa_1, \kappa_2)$ -moral problem in  $\mathbf{m}_{\mathbf{s}}$  has a solution.

*Proof.* As in 3.1, 3.5(1) recalling

- ▮ if  $M_s^t$  for  $s \in I, t \in \{1, 2\}$  are  $\tau$ -models,  $|\tau| \leq \mu, D$  a  $\mu$ -regular filter on  $I$  and  $M_s^1, M_s^2$  are elementarily equivalent, then  $N_1 = \prod_{s \in I} M_s^1/D, N_2 = \prod_{s \in I} M_s^2/D$  are  $\mathbb{L}_{\mu^+, \mu^+}$ -equivalent (and more, see Kennedy-Shelah [KeSh:769], [KeSh:852] and Kennedy-Shelah-Vaananen [KShV:912] on the subject).

□<sub>4.4</sub>



**Observation 4.5.** Assume  $\bar{\mathbf{a}}$  is a  $T_{\text{ord}} - (\kappa_1, \kappa_1)$ -moral problem for  $\mathbf{m}$  so (4.3(6))  $I_\ell = I_\ell(\kappa_\ell)$  for  $\ell = 1, 2$ .

1) If  $I'_1 \subseteq I_1$  is cofinal in  $I_1$  and  $I'_2 \subseteq I_2$  is co-initial in  $I_2$  then  $\bar{\mathbf{a}}$  has a solution in  $\mathbf{m}$  iff  $\bar{\mathbf{a}} \upharpoonright (I'_1 + I'_2) = \langle \mathbf{a}_{s,t} : s <_I t \text{ and } s, t \in I'_1 + I'_2 \rangle$  has a solution in  $\mathbf{m}$ .

1A) Also, above, if  $\bar{\mathbf{b}}$  is a solution of  $\bar{\mathbf{a}}$  in  $\mathbf{m}$  then  $\bar{\mathbf{b}} \upharpoonright (I'_1 + I'_2)$  is a solution of  $\bar{\mathbf{a}}'$ .

1B) Also above, if  $\bar{\mathbf{b}};$  is a solution of  $\bar{\mathbf{a}}'$ , then  $\bar{\mathbf{b}}$  is a solution of  $\bar{\mathbf{a}}$  when:

(a) if  $s \in I_1$  and  $t \in I'_1$  is minimal such that  $s \leq_I t$  then  $\mathbf{b}_s = \mathbf{b}'_t \cap \mathbf{a}_{s,t}$  if  $s <_I t$  and  $\mathbf{b}_s = \mathbf{b}'_t$  if  $s = t$

(b) like (a) replacing  $I_1, I'_1, s <_I t, \mathbf{a}_{s,t}$  by  $I_2, I'_2, t \leq_I s, \mathbf{a}_{t,s}$ .

2) If  $\bar{\mathbf{b}}$  is a solution of  $\bar{\mathbf{a}}$  in  $\mathbf{m}$  and  $\mathbf{b}'_s \in D \wedge \mathbf{b}'_s \leq \mathbf{b}_s$  for  $s \in I_1 + I_2$  then  $\langle \mathbf{b}'_s : s \in I \rangle$  is a solution of  $\bar{\mathbf{a}}$  for  $\mathbf{m}$ .

*Proof.* 1) Easy using the proofs of 3.5, 4.4 or using (1A),(1B).

1A), 1B), 2) Check. □<sub>4.5</sub>

A key point in the inductive construction is:.

**Claim 4.6.** There is no solution to  $\bar{\mathbf{a}}$  in  $\mathbf{m}_\delta$  when:

(a)  $\bar{\mathbf{m}} = \langle \mathbf{m}_\alpha : \alpha \leq \delta \rangle \in S_{\text{ba}}$

(b)  $\bar{\mathbf{a}}$  is a  $T_{\text{ord}} - (\kappa_1, \kappa_2)$ -moral problem in  $\mathbf{m}_0$

(c) if  $\alpha < \delta$  then  $\bar{\mathbf{a}}$  has no solution in  $\mathbf{m}_\alpha$

(d)  $\text{cf}(\delta) \neq \kappa_1$  or  $\text{cf}(\delta) \neq \kappa_2$ .

*Proof.* Let  $\mathbf{m}_\gamma = (\mathfrak{B}_\gamma, D_\gamma)$  for  $\gamma \leq \delta$ ; by symmetry without loss of generality  $\text{cf}(\delta) \neq \kappa_1$  and toward contradiction assume  $\bar{\mathbf{b}} = \langle \mathbf{b}_s : s \in I_1 + I_2 \rangle$  is a solution of  $\bar{\mathbf{a}}$  in  $\mathbf{m}_\delta$ .

Hence  $\mathbf{b}_s \in D$ . Now  $D_\delta$  is not necessarily equal to  $\bigcup_{\gamma < \delta} D_\delta$  but recalling 4.2(3)(d)( $\beta$ )

and  $\langle D_\gamma : \gamma < \delta \rangle$  being increasing, clearly every member of  $D_\delta$  is above some member of  $\bigcup_{\gamma < \delta} D_\gamma$ .

So by Observation 4.5(2) without loss of generality  $s \in I_1 + I_2 \Rightarrow \mathbf{b}_s \in \bigcup_{\gamma < \delta} D_\gamma \subseteq$

$\bigcup_{\gamma < \delta} \mathfrak{B}_\gamma$ .

As  $\text{cf}(\delta) \neq \kappa_1$ , for some  $\gamma < \delta$  we have  $\kappa_1 = \sup\{\alpha < \kappa_1 : \mathbf{b}_{(1,\alpha)} \in \mathfrak{B}_\gamma\}$ , i.e.  $\{s \in I_1 : \mathbf{b}_s \in \mathfrak{B}_\gamma\}$  is co-final in  $I_1$ . So by 4.5(1) without loss of generality

(a)  $s \in I_1 \Rightarrow \mathbf{b}_s \in \mathfrak{B}_\gamma$ .

As  $D_\gamma = D_\delta \cap \mathfrak{B}_\gamma$  by 4.2(2)(c) clearly

(b)  $s \in I_1 \Rightarrow \mathbf{b}_s \in D_\gamma$ .

For  $t \in I_2$  let  $\mathbf{b}'_t = \min\{\mathbf{b} \in \mathfrak{B}_\gamma : \mathfrak{B}_\delta \models \mathbf{b}_t \leq \mathbf{b}\}$ , well defined because  $\mathfrak{B}_\gamma$  is complete.

Now

(c)  $\mathbf{b}'_t \in D_\gamma$  for  $t \in I_2$ .

[Why? Clearly  $\mathbf{b}_t \in \mathfrak{B}_\delta$  as  $\bar{\mathbf{b}}$  is a solution of  $\bar{\mathbf{a}}$  in  $\mathbf{m}_\delta$  and  $\mathbf{b}_t \leq \mathbf{b}'_t, \mathbf{b}'_t \in \mathfrak{B}_\gamma$  by its choice. Also  $\mathbf{b}'_t \in D_\delta$  because  $\mathbf{b}_t \leq \mathbf{b}'_t \cap \mathbf{b}_t \in D_\delta$  and  $D_\delta$  is a filter on  $\mathfrak{B}_\delta$  and lastly  $\mathbf{b}'_t \in D_\gamma$  as  $D_\gamma = D_\delta \cap \mathfrak{B}_\gamma$ .]

(d) if  $s \in I_1, t \in I_2$  then  $\mathbf{b}_s \cap \mathbf{b}'_t \leq \mathbf{a}_{s,t}$ .

[Why? Note  $\mathfrak{B}_\delta \models \mathbf{b}_s \cap \mathbf{b}_t \leq \mathbf{a}_{s,t}$  because  $\bar{\mathbf{b}}$  is a solution of  $\bar{\mathbf{a}}$  in  $\mathfrak{B}_\delta$  hence  $\mathbf{b}_t \leq \mathbf{a}_{s,t} \cup (1 - \mathbf{b}_s)$  and the later  $\in \mathfrak{B}_\gamma$ . So by the choice of  $\mathbf{b}'_t, \mathbf{b}'_t \leq \mathbf{a}_{s,t} \cup (1 - \mathbf{b}_s)$  hence  $\mathbf{b}_s \cap \mathbf{b}'_t \leq \mathbf{a}_{s,t}$ .]

(e)  $\langle \mathbf{b}_s : s \in I_1 \rangle \wedge \langle \mathbf{b}'_t : t \in I_2 \rangle$  solves  $\bar{\mathbf{a}}$  in  $\mathfrak{B}_\gamma$ .

[Why? By (a) + (b) + (c) + (d).]

But this contradicts an assumption. □<sub>4.6</sub>

**Definition 4.7.** Assume  $\mathbf{m} \in K_\kappa^{\text{ba}}$  and  $\bar{\mathbf{a}}$  is a  $(\kappa_1, \kappa_2)$ -moral problem in  $\mathbf{m}$ . We say  $\mathbf{n}$  is a simple  $\bar{\mathbf{a}}$ -solving extension of  $\mathbf{m}$  when:

- (a)  $\mathfrak{B}_\mathbf{n}$  is the completion of  $\mathfrak{B}_\mathbf{n}^\circ$  where
- (b)  $\mathfrak{B}_\mathbf{n}^\circ$  is generated by  $\mathfrak{B}_\mathbf{m} \cup \{y_s : s \in I(\kappa_1, \kappa_2)\}$  freely except the equations which holds in  $\mathfrak{B}_\mathbf{m}$  and  $\Gamma_{\bar{\mathbf{a}}} = \{y_{s_1} \cap y_{s_2} \leq \mathbf{a}_{s_1, s_2} : s_1 \in I_1(\kappa_1) \text{ and } s_2 \in I_2(\kappa_2)\}$
- (c)  $D_\mathbf{n}$  is the filter on  $\mathfrak{B}_\mathbf{n}$  generated by  $D_\mathbf{m} \cup \{y_s : s \in I(\kappa_1, \kappa_2)\}$ .

*Remark 4.8.* We return to this more generally in §5.

**Claim 4.9.** Assume  $\bar{\mathbf{a}}$  is a  $T_{\text{ord}} - (\kappa_1, \kappa_2)$ -moral problem in  $\mathbf{m} \in K_\kappa^{\text{ba}}$  and<sup>6</sup>  $\kappa = \text{cf}(\kappa) > \kappa_1 + \kappa_2$ .

- 1) There is  $\mathbf{n} \in K_{\text{ba}}$  which is a simple  $\bar{\mathbf{a}}$ -solving extension of  $\mathbf{m}$ , unique up to isomorphism over  $\mathfrak{B}_\mathbf{m}$ .
- 2) Above  $\mathbf{m} \leq_\kappa^{\text{ba}} \mathbf{n}$  (so  $\mathbf{n} \in K_\kappa^{\text{ba}}$ ).
- 3) If  $\bar{\mathbf{a}}^*$  is a  $T_{\text{ord}} - (\theta_1, \theta_2)$ -moral problem of  $\mathbf{m}$  with no solution in  $\mathbf{m}$  and  $\theta_1 \notin \{\kappa_1, \kappa_2\}$  or  $\theta_2 \notin \{\kappa_1, \kappa_2\}$  then  $\bar{\mathbf{a}}^*$  has no solution in  $\mathbf{n}$ .

*Proof.* 1) Let  $I_\ell = I_\ell(\kappa_\ell)$  for  $\ell = 1, 2$  and  $I = I_1 + I_2$ .

First

(\*)<sub>1</sub> the set of equations  $\Gamma_{\bar{\mathbf{a}}}$  is finitely satisfiable in  $\mathfrak{B}_\mathbf{m}$ .

Why? Let  $0 = 0_{\mathfrak{B}_\mathbf{m}}$ . We prove two stronger statements (each implying (\*)<sub>1</sub>).

(\*)<sub>1.1</sub> if  $t_1 \in I_1$  then we can find  $\langle \mathbf{b}'_s : s \in I \rangle \in {}^I \mathfrak{B}$  such that:

- (a)  $\mathbf{b}'_s \in D_\mathbf{m} \subseteq \mathfrak{B}_\mathbf{m}$  if  $(s <_{I_1} t_1) \vee (s \in I_2)$
- (b) if  $s_1 \in I_1, s_2 \in I_2$  then  $\mathbf{b}'_{s_1} \cap \mathbf{b}'_{s_2} \leq \mathbf{a}_{s_1, s_2}$ .

[Why? Let  $\mathbf{b}'_s$  be:

- $\mathbf{a}_{s, t_1}$  if  $s <_I t_1$  (so  $s \in I_1$ )
- $\mathbf{a}_{t_1, s}$  if  $s \in I_2$
- $0_\mathfrak{B}$  if  $t_1 \leq_I s \in I_1$ .

---

<sup>6</sup>It seems that  $\min\{\kappa_1, \kappa_2\} < \kappa$  suffice; the only difference in the proof is in proving (\*)<sub>5</sub>.

Now clause (a) is obvious and as for clause (b), let  $s_1 \in I_1, s_2 \in I_2$ , now if  $t_1 \leq_I s_1 \in I_2$  then  $\mathbf{b}'_{s_1} \cap \mathbf{b}'_{s_2} = 0 \cap \mathbf{b}'_{s_2} = 0 \leq \mathbf{a}_{s_1, s_2}$  and if  $s <_I t_1$  then  $\mathbf{b}'_{s_1} \cap \mathbf{b}'_{s_2} = \mathbf{a}_{s_1, t} \cap \mathbf{a}_{t, s_2}$  which is  $\leq \mathbf{a}_{s_1, s_2}$  by 4.3(1)(d),(e).]

(\*)<sub>1,2</sub> if  $t_2 \in I_2$  then we can find  $\langle \mathbf{b}'_s : s \in I \rangle \in {}^I \mathfrak{B}$  such that

(a)  $\mathbf{b}'_s \in D_{\mathbf{m}} \subseteq \mathfrak{B}_{\mathbf{m}}$  if  $s \in I_1$  or  $t_2 <_{I_2} s$

(b) if  $s_1 \in I_2, s_2 \in I_2$  then  $\mathbf{b}'_{s_1} \cap \mathbf{b}'_{s_2} \leq \mathbf{a}_{s_1, s_2}$ .

[Why? Similarly.]

Now (\*)<sub>1</sub> is easy: if  $\Gamma' \subseteq \Gamma_{\bar{\mathbf{a}}}$  is finite let  $t_* \in I_1$  be such that: if  $t \in I_1$  and  $y_t$  appears in  $\Gamma'$  then  $t \leq_I t_*$ . Choose  $\langle \mathbf{b}'_s : s \in I \rangle$  as in (\*)<sub>1,1</sub> for  $t_*$  and let  $h$  be the function  $y_s \mapsto \mathbf{b}'_s$  for  $s \in I$ . Now think.

Clearly it follows by (\*)<sub>1</sub> that

(\*)<sub>2</sub> (a) there is a Boolean Algebra  $\mathfrak{B}_{\mathbf{n}}^0$  extending  $\mathfrak{B}_{\mathbf{m}}$  as described in clause (b) of Definition 4.7

(b) there is a Boolean Algebra  $\mathfrak{B}_{\mathbf{n}}$  as described in (a) of Definition 4.7: the completion of  $\mathfrak{B}_{\mathbf{n}}^0$

(c)  $D_{\mathbf{n}}$  is chosen as the filter on  $\mathfrak{B}_{\mathbf{n}}$  generated by  $D_{\mathbf{m}} \cup \{y_s : s \in I\}$  satisfies  $D_{\mathbf{m}} = D_{\mathbf{n}} \cap \mathfrak{B}_{\mathbf{m}}$ , in particular  $0 \notin D_{\mathbf{n}}$

(d)  $\mathfrak{B}_{\mathbf{n}}$  satisfies the  $\kappa$ -c.c.

[Why? Clauses (a),(b) follows by (\*)<sub>1</sub> and for clauses (c),(d) see (\*)<sub>4</sub> and (\*)<sub>5</sub> in the proof of (2), respectively; in particular  $0 \notin D_{\mathbf{n}}$ .]

Together we have  $\mathbf{n} = (\mathfrak{B}_{\mathbf{n}}, D_{\mathbf{n}}) \in K_{\text{ba}}$ , as for  $\mathbf{m} \leq_{\text{ba}} \mathbf{n}$ , see part (2).

2) Now (by part (1)) we have  $\mathfrak{B}_{\mathbf{m}} \subseteq \mathfrak{B}_{\mathbf{n}}$ , but moreover)

(\*)<sub>3</sub>  $\mathfrak{B}_{\mathbf{m}} < \mathfrak{B}_{\mathbf{n}}$ .

[Why? If not, then some  $\mathbf{d} \in \mathfrak{B}_{\mathbf{n}} \setminus \{0\}$  is disjoint to  $\mathbf{b}$  for a dense subset of  $\mathbf{b} \in \mathfrak{B}_{\mathbf{m}} \setminus \{0\}$ . Let  $\mathbf{d} = \sigma(y_{s_0}, \dots, y_{s_{n-1}}, \bar{c})$  where  $\sigma$  is a Boolean term,  $s_0 <_I \dots <_I s_{n-1}$  and  $\bar{c}$  is from  $\mathfrak{B}_{\mathbf{m}}$ . We may replace  $\mathbf{d}$  by any  $\mathbf{d}' \in \mathfrak{B}^+$  satisfying  $\mathbf{d}' \leq_{\mathfrak{B}} \mathbf{d}$ . Without loss of generality  $d = \cap \{y_{s_\ell}^{\text{if}(\eta(\ell))} : \ell < n\} \cap c > 0$  where  $c \in \mathfrak{B}_{\mathbf{m}}, \eta(\ell) \in \{0, 1\}$  for  $\ell < n$  and without loss of generality for every  $\ell, k < n$  we have  $s_\ell \in I_1 \wedge s_k \in I_2 \Rightarrow (c \leq \mathbf{a}_{s_\ell, s_k}) \vee (c \cap \mathbf{a}_{s_\ell, s_k} = 0)$ .

We now define a function  $h$  from  $\{y_s : s \in I\}$  into  $\mathfrak{B}_{\mathbf{m}}$  as follows:  $h(y_s)$  is:

- <sub>1</sub>  $c$  if  $s = s_\ell \wedge \eta(\ell) = 1$
- <sub>2</sub>  $0$  if otherwise.

Now

- <sub>3</sub> if  $t_1 \in I_1, t_2 \in I_2$  then  $\mathfrak{B}_{\mathbf{m}} \models "h(y_{t_1}) \cap h(y_{t_2}) \leq \mathbf{a}_{t_1, t_2}"$ .

[Why? If  $h(t_1) = 0 \vee h(t_2) = 0$  this is obvious, otherwise for some  $\ell(1) < \ell(2) < n$  we have  $t_1 = s_{\ell(1)}, t_2 = s_{\ell(2)}$  and  $\eta(\ell(1)) = 1 = \eta(\ell(2))$ . So we have to prove  $c = c \cap c \leq \mathbf{a}_{t_1, t_2}$  but otherwise by the choice of  $c, c \cap \mathbf{a}_{t_1, t_2} = 0$ , hence recalling 4.7(b) we have  $\mathfrak{B}_{\mathbf{n}} \models "y_{s_1} \cap y_{s_2} \cap c = 0"$  contradiction to our current assumption  $\mathfrak{B}_{\mathbf{n}} \models "d > 0"$ ; so •<sub>3</sub> holds indeed.]

By the choice of  $\mathfrak{B}_{\mathbf{n}}$  recalling  $\mathfrak{B}_{\mathbf{m}}$  is complete, by the choice of  $h$  and •<sub>3</sub> there is a projection  $\hat{h}$  from  $\mathfrak{B}_{\mathbf{n}}$  onto  $\mathfrak{B}_{\mathbf{m}}$  extending  $h$ , so clearly  $\hat{h}(d) = c$  and this implies

$c_1 \in \mathfrak{B}_m \wedge 0 < c_1 \leq c \Rightarrow \mathfrak{B}_n \models "c_1 \cap d \geq 0"$  contradicting the choice of  $d$ . So indeed  $(*)_3$  holds.]

$$(*)_4 \ D_m = D_n \cap \mathfrak{B}_m.$$

[Why? Otherwise there are  $c_1 \in D_m, c_2 \in \mathfrak{B}_m \setminus D_m$  and  $s_0 <_I \dots <_I s_{n-1}$  such that  $\mathfrak{B}_n \models " \bigcap_{\ell < n} y_{s_\ell} \cap c_1 \leq c_2 "$ . As  $\mathbf{a}_{t_1, t_2} \in D_m$  for  $t_1 <_I t_2$ , without loss of generality  $c_1 \leq \mathbf{a}_{s_\ell, s_k}$  for  $\ell < k < n$ .

Now letting  $c = c_1 - c_2$  we continue as in the proof of  $(*)_3$  defining  $h, \hat{h}$  and apply the projection  $\hat{h}$  to  $" \bigcap_{\ell < n} y_{s_\ell} \cap c_1 \leq c_2 "$ .]

$$(*)_5 \ \mathfrak{B}_n \text{ satisfies the } \kappa\text{-c.c..}$$

[Why? If not, then there are pairwise disjoint, positive  $d_i \in \mathfrak{B}_n$  for  $i < \kappa$ . So as in the proof of  $(*)_3$ , without loss of generality  $d_i = \bigcap \{ y_{s(i, \ell)}^{\text{if}(\mathbf{t}(i, \ell))} : \ell < n(i) \} \cap c_i$  where  $c_i \in \mathfrak{B}_m, \mathbf{t}(i, \ell) \in \{0, 1\}$  and  $s(i, 0) <_I s(i, 1) <_I \dots <_I s(i, n(i) - 1)$ .

Again as there, without loss of generality for every  $\ell < k < n(i)$  we have  $(\mathbf{a}_{s(i, \ell), s(i, k)} \leq c_i) \vee (\mathbf{a}_{s(i, \ell), s(i, k)} \cap c_i = 0)$  so  $\mathbf{t}(i, \ell) = 1 = \mathbf{t}(i, k) \wedge \ell < k \Rightarrow c_i \leq \mathbf{a}_{s(i, \ell), s(i, k)}$ .

As  $\kappa = \text{cf}(\kappa) > \kappa_1 + \kappa_2$  by an assumption of 4.9 without loss of generality  $n(i) = n, \mathbf{t}(i, \ell) = \mathbf{t}(\ell)$  and  $s(i, \ell) = s_\ell$  for  $i < \kappa, \ell < n$  and as  $\mathfrak{B}_m$  satisfies the  $\kappa$ -c.c. we can find  $i < j < \kappa$  such that  $\mathfrak{B}_m \models "0 < c_i \cap c_j"$  and let  $c = c_i \cap c_j$  so we continue as before.]

So together by  $(*)_3, (*_4), (*_5)$  we have  $\mathbf{m} \leq_\kappa^{\text{ba}} \mathbf{n} \in K_\kappa^{\text{ba}}$  as promised.

3) Let  $I^* = I(\theta_1, \theta_2), I_1^* = I_1(\theta_1), I_2^* = I_2(\theta_2)$  and recall  $\bar{\mathbf{a}}^* = \langle \mathbf{a}_{s, t}^* : s <_{I^*} t \rangle$  is a  $T_{\text{ord}} - (\theta_1, \theta_2)$ -moral problem in  $\mathbf{m}$ . Toward contradiction assume  $\bar{\mathbf{b}} = \langle \mathbf{b}_t : t \in I^* \rangle$  solve the problem  $\bar{\mathbf{a}}^*$  in  $\mathbf{n}$  so let  $\mathbf{b}_t = \sigma_t(y_{s(t, 0)} \dots, y_{s(t, n(t)-1)}, c_{t, 0}, \dots, c_{t, m(t)-1})$  with  $c_{t, k} \in \mathfrak{B}_m, s(t, \ell) \in I$  and without loss of generality  $s(t, \ell) <_I s(t, \ell + 1)$  for  $\ell < n(t) - 1$  so  $s(t, \ell) \in I$ .

By symmetry without loss of generality

$$(*)_6 \ \theta_1 \notin \{\kappa_1, \kappa_2\}.$$

Recalling 4.5, we can replace  $\mathbf{b}_t$  by any  $\mathbf{b}'_t \leq \mathbf{b}_t$  which is from  $D_n$ , so as  $\bigwedge_\ell y_{s(t, \ell)} \in D_n$ , without loss of generality  $\ell < n(t) \Rightarrow \mathbf{b}_t \leq y_{s(t, \ell)}$ , so without loss of generality  $\mathbf{b}_t = \bigcap \{ y_{s(t, \ell)} : \ell < n(t) \} \cap c_t$  for some  $c_t \in D_m$  recalling  $D_m = D_m \cap \mathfrak{B}_m$ .

By the  $\Delta$ -system lemma (recalling 4.5(1)) without loss of generality

- $\oplus$  if  $\theta_1 > \aleph_0$  then
  - (a)  $t \in I_1^* \Rightarrow n(t) = n(*)$
  - (b) if  $t \in I_1^*$  then  $s(t, \ell) \in I_1^* \Leftrightarrow \ell < \ell(*)$
  - (c)  $\langle \langle s(t, \ell) : \ell < n(*) \rangle : t \in I_1^* \rangle$  is an indiscernible sequence in the linear order  $I$ , for quantifier free formulas.

But we shall not use it. As  $\theta_1 \neq \kappa_1, \kappa_2$  it follows that for some  $s_1^\circ, s_2^\circ$

$$(*)_7 \ s_1^\circ \in I_1, s_2^\circ \in I_2 \text{ and } s(t, \ell) \notin [s_1^\circ, s_2^\circ]_I \text{ for every } t \in I_1^*, \ell < n(t).$$

Again by 4.5(2) without loss of generality

$$(*)_8 \text{ if } t \in I_2^* \text{ then } \mathbf{b}_t \leq y_{s_1^\circ} \cap y_{s_2^\circ}.$$

We now define a function  $h$  from  $\{y_s : s \in I\}$  into  $\mathfrak{B}_n$ , (yes! not  $\mathfrak{B}_m$ ) by:

- (\*)<sub>9</sub>  $h(y_s)$  is:
- $\mathbf{a}_{s,s_1^\circ} \cap \mathbf{a}_{s_1^\circ,s_2^\circ}$  if  $s <_I s_1^\circ$
  - $\mathbf{a}_{s_1^\circ,s} \cap y_s \cap \mathbf{a}_{s,s_2^\circ}$  if  $s \in I, s_1^\circ \leq_I s \leq_I s_2^\circ$
  - $\mathbf{a}_{s_1^\circ,s_2^\circ} \cap \mathbf{a}_{s_2^\circ,s}$  if  $s_2^\circ <_I s$ .

Note

- (\*)<sub>10</sub>  $h(y_s) \in D_n$  for  $s \in I$ .

[Why? Because  $\mathbf{a}_{s,t} \in D$  for  $s <_J t$  and  $y_s \in D$  for  $s \in I$ .]

- (\*)<sub>11</sub>  $h(y_{s_1}) \cap h(y_{s_2}) \leq \mathbf{a}_{s_1,s_2}$  for  $s_1 \in I_1, s_2 \in I_2$ .

[Why? If  $s_1, s_2 \in [s_1^\circ, s_2^\circ]_I$  this holds by the definition of  $\mathfrak{B}_n$ , i.e. as  $h(s_1) \leq y_{s_1}, h(s_2) \leq y_{s_2}$  and  $\mathfrak{B}_n \models "y_{s_1} \cap y_{s_2} \leq \mathbf{a}_{s_1,s_2}"$ .

If  $s_1 <_{I^*} s_1^\circ \wedge s_2^\circ <_{I^*} s_2$  then (\*)<sub>11</sub> says:  $\mathbf{a}_{s_1,s_1^\circ} \cap \mathbf{a}_{s_1^\circ,s_2^\circ} \cap \mathbf{a}_{s_2^\circ,s_2} \leq \mathbf{a}_{s_1,s_2}$  which obviously holds (as  $\bar{\mathbf{a}}$  is a  $T_{\text{ord}}$ -morality problem).

If  $s_1 <_{I^*} s_1^\circ \wedge s_2 \in [s_1^\circ, s_2^\circ]_{I^*}$  then this means:  $(\mathbf{a}_{s_1,s_1^\circ} \cap \mathbf{a}_{s_1^\circ,s_2^\circ}) \cap (\mathbf{a}_{s_1^\circ,s_2} \cap y_{s_2} \cap \mathbf{a}_{s_2,s_2^\circ}) \leq \mathbf{a}_{s_1,s_2}$ ; but as we have  $\mathbf{a}_{s_1,s_1^\circ} \cap \mathbf{a}_{s_1^\circ,s_2} \leq \mathbf{a}_{s_1,s_2}$  this holds.

If  $s_1 \in [s_1^\circ, s_2^\circ]_{I^*}$  and  $s_2^\circ <_{I^*} s_2$  this means  $(\mathbf{a}_{s_1^\circ,s_1} \cap y_{s_1} \cap \mathbf{a}_{s_1,s_2}) \cap (\mathbf{a}_{s_1^\circ,s_2^\circ} \cap \mathbf{a}_{s_2^\circ,s_2}) \leq \mathbf{a}_{s_1,s_2}$  which holds for similar reasons. So (\*)<sub>11</sub> holds indeed.]

By the choice of  $\mathfrak{B}_n^\circ$  and  $\mathfrak{B}_n$  there is a homomorphism  $\hat{h}$  from  $\mathfrak{B}_n$  into  $\mathfrak{B}_m$ , extending  $\text{id}_{\mathfrak{B}_m}$  and extending  $h$ . Now easily  $\hat{h}(\mathbf{b}_t) \in D$  for  $t \in I^*$  because  $\mathbf{b}_t = \cap \{y_{s(t,\ell)} : \ell < n(t)\} \cap c_t, c_t \in D_m$  hence  $\hat{h}(c_t) = c_t \in D_m$  and by (\*)<sub>10</sub> we have  $\hat{h}(y_{s(t,\ell)}) \in D_m$ .

Now  $\langle \hat{h}(\mathbf{b}_t) : t \in I^* \rangle$  still form a solution of  $\bar{\mathbf{a}}^*$  and by (\*)<sub>7</sub> we have  $t \in I_1^* \Rightarrow h(\mathbf{b}_t) \in \mathfrak{B}_m$  hence without loss of generality:

- (\*)<sub>12</sub>  $t \in I_1^* \Rightarrow \mathbf{b}_t \in \mathfrak{B}_m$ .

Now define  $\mathbf{b}'_t$  for  $t \in I^*$  by:  $\mathbf{b}'_t$  is:

- $\mathbf{b}_t$  if  $t \in I_1^*$
- $c_t$  if  $t \in I_2^*$ .

It suffices to prove that  $\langle \mathbf{b}'_t : t \in I^* \rangle$  solves  $\bar{\mathbf{a}}^*$  in  $\mathbf{m}$ . Clearly  $t \in I^* \Rightarrow \mathbf{b}'_t \in D_m$ , so let  $t_1 \in I_1^*, t_2 \in I_2^*$ . We have to prove that  $\mathbf{b}'_{t_1} \cap \mathbf{b}'_{t_2} \leq \mathbf{a}_{t_1,t_2}$  but we know only  $\mathbf{b}_{t_1} \cap \mathbf{b}_{t_2} \leq \mathbf{a}_{t_1,t_2}$  which means  $\mathbf{a}_{t_1,t_2} \geq \mathbf{b}'_{t_1} \cap (\bigcap_{\ell < n(t_2)} y_{s(t_2,\ell)} \cap c_{t_2}) = (\mathbf{b}'_{t_1} \cap \mathbf{b}'_{t_2}) \cap \bigcap \{y_{s(t,\ell)} : \ell < n(t_2)\}$ .

$\bigcap \{y_{s(t,\ell)} : \ell < n(t_2)\}$ .

Let  $h_{t_2}$  be a projection from  $\mathfrak{B}_n$  onto  $\mathfrak{B}_m$  such that  $h_{t_2}(y_{s(t_2,\ell)}) = c_t$  if  $\ell < n(t)$  and  $h_{t_2}(y_s) = 0$  if  $s \in I \setminus \{s(t_2,\ell) : \ell < n(t_2)\}$ , as earlier it exists and applying it we get the desired inequality.  $\square_{4.9}$

**Theorem 4.10.** *For any  $\lambda$  and regular*

- (a)  $\theta_1, \theta_2 \leq \lambda$  such that  $\theta_1 + \theta_2 > \aleph_0$  there is a regular filter  $D$  on  $\lambda$  such that: for every dense linear order  $M$ , in  $M^\lambda/D$  there is a  $(\theta_1, \theta_2)$ -pre-cut but no  $(\kappa_1, \kappa_2)$ -pre-cut when  $\kappa_1, \kappa_2$  are regular  $\leq \lambda$  and  $\{\theta_1, \theta_2\} \not\subseteq \{\kappa_1, \kappa_2\}$
- (b) if  $M$  is  $(^{>2, <})^\lambda/D$  then  $\text{t}(M) \geq \lambda^+$ .

*Remark 4.11.* 1) Why do we need  $\theta_1 + \theta_2 > \aleph_0$ ? To prove  $(*)_1$ .  
 2) In fact, this demand is necessary, see 4.13 below.

*Proof.* Let  $\kappa = \lambda^+$ .

$(*)_1$  there is  $\mathbf{m}_0 \in K_\kappa^{\text{ba}}$  and a  $T_{\text{ord}} - (\theta_1, \theta_2)$ -moral problem not solved in it.

[Why? By [Sh:c, Ch.VI, §3] there is an ultrafilter  $D$  on  $\lambda$  such that in  $(\mathbb{Q} <)^\lambda / D$  there is a  $(\theta_1, \theta_2)$ -cut. Define  $\mathbf{m}$  by  $\mathfrak{B}_{\mathbf{m}} = \mathcal{P}(\lambda)$ ,  $D_{\mathbf{m}} = D$ .]

Let  $\langle W_\alpha : \alpha < 2^\lambda \rangle$  be a partition of  $2^\lambda$  to sets each of cardinality  $2^\lambda$  such that  $W_\alpha \cap \alpha = \emptyset$ .

$(*)_2$  we can choose  $\mathbf{m}_\alpha$  and  $\langle \bar{\mathbf{a}}_\gamma : \gamma \in W_\alpha \rangle$  by induction on  $\alpha \leq 2^\lambda$  such that:

- (a)  $\mathbf{m}_\alpha \in K_\kappa^{\text{ba}}$  has cardinality  $\leq 2^\lambda$
- (b)  $\langle \mathbf{m}_\beta : \beta \leq \alpha \rangle \in S_\kappa^{\text{ba}}$
- (c)  $\mathbf{m}_0$  is as in  $(*)_1$
- (d)  $\langle \bar{\mathbf{a}}_\gamma : \gamma \in W_\gamma \rangle$  be such that  $\bar{\mathbf{a}}_\gamma$  is a  $T_{\text{ord}} - (\kappa_{\gamma,1}, \kappa_{\gamma,2})$ -morality problem in  $\mathbf{m}_\alpha$  and  $\kappa_{\gamma,1}, \kappa_{\gamma,2}$  are regular  $\leq \lambda$  and  $\{\theta_1, \theta_2\} \not\subseteq \{\kappa_{\gamma,1}, \kappa_{\gamma,2}\}$  and any such  $\bar{\mathbf{a}}$  appears in the sequence
- (e) if  $\alpha = \gamma + 1$  then  $\gamma \in W_\beta$  for some  $\beta \leq \alpha$  and in  $\mathbf{m}_\alpha$  there is a solution for  $\bar{\mathbf{a}}_\gamma$
- (f) in  $\mathbf{m}_\alpha$  there is no solution to  $\bar{\mathbf{a}}^*$ .

[Why we can?

Now for  $\alpha = 0$  use  $(*)_1$ , for  $\alpha$  limits use 4.6 and for  $\alpha$  successor use 4.9.]

$(*)_4$  letting  $\mathbf{m} = \mathbf{m}_{2^\lambda}$  we have  $\mathfrak{B}_{\mathbf{m}} = \cup \{ \mathfrak{B}_{\mathbf{m}_\alpha} : \alpha < 2^\lambda \}$  and  $D_{\mathbf{m}} = \cup \{ D_{\mathbf{m}_\alpha} : \alpha < 2^\lambda \}$ .

[Why? Because  $\langle \mathbf{m}_\alpha : \alpha \leq 2^\lambda \rangle \in S_\kappa^{\text{ba}}$  and  $\text{cf}(2^\lambda) \geq \kappa$ .]

$(*)_5$  there is a regular excellent filter  $D_0$  on  $\lambda$  and homomorphism  $\mathbf{j}$  from  $\mathcal{P}(\lambda)$  onto  $\mathfrak{B}_{\mathbf{m}}$ .

[Why? See [MiSh:997].]

$(*)_6$  let  $D = \mathbf{j}^{-1}(D_{\mathbf{m}})$ .

So  $D$  is a filter on  $\lambda$ , and by 3.8 for  $\theta = \aleph_0$  (or Malliaris-Shelah [MiSh:997]) we are done.  $\square_{4.10}$

**Conclusion 4.12.** *If  $\lambda \geq \aleph_2$  the results of Malliaris-Shelah [MiSh:998] cannot be generalized to reduced powers (quantifier free types, of course).*

*Proof.* Choose in 4.10 the pair  $(\theta_1, \theta_2)$  as  $(\aleph_1, \aleph_2)$ .  $\square_{4.12}$

**Observation 4.13.** *If  $\mathbf{m} \in K_\kappa^{\text{ba}}$  then any  $T_{\text{ord}} - (\aleph_0, \aleph_0)$ -morality problem  $\bar{\mathbf{a}}$  has a solution.*

*Proof.* Let  $\mathbf{b}_{(1,n)} = \mathbf{b}_{(2,n)} = \mathbf{b}_n := \cap \{ \mathbf{a}_{(1,\ell),(2,k)} : \ell, k \leq n \}$ , clearly  $\mathbf{b}$  for  $s \in I(\aleph_0, \aleph_0) \Rightarrow \mathbf{b}_s \in D$  and  $(s, t) \in I(1, \aleph_0) \times I(2, \aleph_0) \Rightarrow \mathbf{b}_s \cap \mathbf{b}_t \leq \mathbf{a}_{s,t}$ .  $\square_{4.13}$

**Claim 4.14.** *In  $M_*^{\mathfrak{B}}/D$ , any increasing sequence of length  $< \kappa^+$  has an upper bound when (A) or (B) holds, where:*

- (A) (a)  $M_* = (\omega^>\mu, \trianglelefteq)$   
 (b)  $\mathfrak{B}$  is a complete Boolean Algebra which is  $(< \theta)$ -distributive  
 (c)  $D$  is a  $(\mu, \theta)$ -regular,  $\theta$ -complete filter on  $A$   
 (d)  $(\mathbb{Q}, <)^{\mathfrak{B}}/D$  has no  $(\sigma, \sigma)$ -pre-cut for any regular  $\sigma \leq \kappa$
- (B) (a) – (c) as above  
 (d) every  $T_{\text{tr}} - (\sigma, \sigma)$ -moral problem in  $\mathbf{m}(\mathfrak{B}, D)$  has a  $T_{\text{tr}} - (\sigma, \sigma)$ -moral solution in  $\mathbf{m}(\mathfrak{B}, D)$  where:
- ( $\alpha$ )  $\bar{\mathbf{a}}$  is a  $T_{\text{tr}}$ -moral problem when:
- $\bar{\mathbf{a}} = \langle \mathbf{a}_{\alpha, \beta} : \alpha < \beta < \sigma \rangle$
  - $\mathbf{a}_{\alpha, \beta} \in D$
  - if  $u \subseteq \sigma$  is finite and  $\mathbf{c} \in \mathfrak{B}^+$  then for some  $\bar{\eta} = \langle \eta_\alpha : \alpha \in u \rangle$  we have  $\eta_\alpha \in |u|^{>|u|}$  for  $\alpha \in u$  and  $\mathbf{c} \leq \mathbf{a}_{\alpha, \beta} \Rightarrow \eta_\alpha \trianglelefteq \eta_\beta$  and  $\mathbf{c} \cap \mathbf{a}_{\alpha, \beta} = 0_{\mathfrak{B}} \Rightarrow \neg(\eta_\alpha \trianglelefteq \eta_\beta)$  for  $\alpha < \beta$  from  $u$
- ( $\beta$ )  $\bar{\mathbf{b}} = \langle \mathbf{b}_\alpha : \alpha < \sigma \rangle$  is a  $T_{\text{tr}} - \sigma$ -solution of  $\bar{\mathbf{a}}$  when  $\mathbf{b}_\alpha \in D$  and  $\mathbf{b}_\alpha \cap \mathbf{b}_\beta \leq \mathbf{a}_{\alpha, \beta}$  for  $\alpha < \beta < \sigma$ .

*Proof.* If clause (A), as in [Sh:a, Ch.VI,2.7] or [MiSh:998].

If clause (B), as above.

□<sub>4.14</sub>

§ 5. THE ORDER  $\leq_{\lambda}^{\text{rp}}$ 

§ 5(A). On  $\leq_{\lambda}^{\text{rp}}$ .

**Convention 5.1.**  $T$  denotes a complete first order theory and, if not said otherwise with elimination of quantifiers.

Recall (from 0.10).

**Definition 5.2.** 1) Let  $T_1 \leq_{\lambda, \mu}^{\text{rp}} T_2$  means: if  $D$  is a regular filter on  $\lambda$  and  $M_{\ell}$  is a model of  $T_{\ell}$  for  $\ell = 1, 2$  and  $(M_2)^{\lambda}/D$  is  $(\mu^+, \text{atomic})$ -saturated, then so is  $(M_1)^{\lambda}/D$ .

1A) Writing  $\triangleleft$  instead of  $\leq$  means that  $(T_1 \leq_{\lambda, \mu}^{\text{rp}} T_2) \wedge \neg(T_2 \leq_{\lambda, \mu}^{\text{rp}} T_1)$ .

2) Omitting  $\mu$  means  $\mu = \lambda$ .

3) Omitting  $\lambda, \mu$  means for every  $\lambda$ , may write  $\leq_{\text{rp}}$ .

4) Similarly for  $(T_1, \Delta_1) \leq_{\lambda, \mu}^{\text{rp}} (T_2, \Delta_2)$ .

5) Let  $(T_2, *) \leq_{\lambda, \mu}^{\text{rp}} (T_2, *)$  means that for every finite set  $\Delta_1$  of atomic formulas of  $\mathbb{L}(\tau(T_2))$ , there is a finite set  $\Delta_2$  of atomic formulas from  $\mathbb{L}(\tau(T_2))$  such that  $(T_1, \Delta_1) \leq_{\lambda, \mu}^{\text{rp}} (T_2, \Delta_2)$ .

Of course

**Fact 5.3.** 1) In 5.2 the choices of  $M_1, M_2$  does not matter.

2)  $\leq_{\lambda, \mu}^{\text{rp}}, \leq_{\lambda}^{\text{rp}}, \leq_{\text{rp}}$  are quasi order.

3) Assume  $M_2 = M_1^{\text{eq}}$ , see [Sh:c, Ch.III] and  $T_{\ell} = \text{Th}(M_{\ell})$  for  $\ell = 1, 2$  and  $T_1$  with elimination of quantifiers, then  $T_1, T_2$  are  $\leq^{\text{rp}}$ -equivalent.

*Proof.* 1) Because in 5.2 the filter  $D$  is regular (see  $\boxplus$  from 4.4).

2) Should be clear. □<sub>5.3</sub>

**Claim 5.4.** If  $T$  has  $\text{SOP}_3$  for quantifier free formulas then  $T$  is  $\leq_{\lambda}^{\text{rp}}$ -maximal.

*Proof.* By §1, §2. □<sub>5.4</sub>

**Definition 5.5.** Assume  $\mathbf{m} \in K_{\text{ba}}$  and  $\bar{\mathbf{a}}$  is a  $\mu$ -morality problem (see 3.7(2); we omit  $\theta$  as  $\theta = \aleph_0$ ).

We say  $\mathbf{n}$  is a simple  $\bar{\mathbf{a}}$ -solving extension of  $\mathbf{m}$  when:

- (a)  $\mathfrak{B}_{\mathbf{n}}$  is the completion of  $\mathfrak{B}_{\mathbf{m}}^0$ , where
- (b)  $\mathfrak{B}_{\mathbf{n}}^0$  is a Boolean Algebra generated by  $\mathfrak{B}_{\mathbf{m}} \cup \{y_{\alpha} : \alpha < \mu\}$  freely except
  - (α) the equations satisfied in  $\mathfrak{B}_{\mathbf{m}}$
  - (β)  $\Gamma_{\bar{\mathbf{a}}} = \{ \bigcap_{\alpha \in u} y_{\alpha} \leq \mathbf{a}_u : u \subseteq \mu \text{ is of cardinality } < \theta \}$ .
- (c)  $D_{\mathbf{n}}$  is the filter of  $\mathfrak{B}_{\mathbf{n}}$  generated by  $\{y_{\alpha} : \alpha < \mu\} \cup D_{\mathbf{m}}$ .

**Claim 5.6.** Assume  $\mathbf{m} \in K_{\kappa}^{\text{ba}}$ .

If (A) then (B) where:

- (A) (a)  $\mathbf{a}$  is a  $T_{\text{ord}} - (\kappa_1, \kappa_2)$ -moral problem in  $\mathbf{m}$
- (b)  $I = I(\kappa_1, \kappa_2)$  and  $\mu = \kappa_1 + \kappa_2 \geq \theta$
- (c) we define  $\bar{\mathbf{a}} = \langle \mathbf{a}'_u : u \in [I(\kappa_1, \kappa_2)]^{<\theta} \rangle$  by  $\mathfrak{B} \models \mathbf{a}'_u = \bigcap \{ \mathbf{a}_{s,t} : s \in I_1(\kappa_1) \cap u \text{ and } t \in I_2(\kappa_1) \}$  problem in  $\mathbf{m}$



- (B) (a)  $\langle \mathbf{a}'_u : u \in [I(\kappa_1, \kappa_2)]^{<\theta} \rangle$  is a  $(\mu, \theta)$ -moral problem in  $\mathbf{m}$ , moreover, it is a  $(\mu, \theta, 1, T_{\text{ord}})$ -model problem
- (b) if  $\theta = \aleph_0$  then  $\mathbf{a}$  has a  $T_{\text{ord}} - (\kappa_1, \kappa_2)$ -moral solution iff  $\mathbf{a}'$  has a  $(\mu, \theta, 1, T_{\text{ord}})$ -moral solution.

*Proof.* As before.  $\square_{5.6}$

**Claim 5.7.** 1) In 5.5,  $\mathbf{n}$  is well defined,  $\in K_{\text{ba}}$  and  $\mathbf{m} \leq \mathbf{n}$ .  
 2) Moreover, if  $\mathbf{m} \in K_{\kappa}^{\text{ba}}, \kappa > \mu$  then  $\mathbf{n} \in K_{\kappa}^{\text{ba}}$  and  $\mathbf{m} \leq_{\kappa}^{\text{ba}} \mathbf{n}$ .

*Proof.* 1) The point

- (\*)<sub>1</sub> if  $u \subseteq \mu$  is finite then there is a function  $h = h_u$  from  $\{y_{\alpha} : \alpha < \mu\}$  into  $\mathfrak{B}_{\mathbf{m}}$  such that:
- ( $\alpha$ ) if  $v \subseteq \mu$  is finite then  $\mathbf{B}_{\mathbf{m}} \models \bigcap_{\alpha \in v} h(y_{\alpha}) \leq \mathbf{a}_u$
- ( $\beta$ ) if  $\alpha \in u$  then  $h(y_{\alpha}) \in D$ .

[Why? Define  $h$  by:  $h(y_{\alpha})$

- $\mathbf{a}_u$  if  $\alpha \in u$
- $0_{\mathfrak{B}_{\mathbf{m}}}$  if  $\alpha \in \mu \setminus u$ .

Now continue as in the proof of 4.9.  $\square_{5.7}$

We like to prove  $\neg(T_1 \leq_{\lambda}^{\text{rp}} T_2)$  for some pairs.

**Claim 5.8.** Assume  $\lambda > \aleph_0$  and  $T_1, T_2$  are complete first order theories with elimination of quantifiers.

1) We have  $\neg(T_1 \leq_{\lambda}^{\text{rp}} T_2)$  iff for some  $\mathbf{m}, \bar{\mathbf{a}}^*$

- (\*) <sub>$\lambda, \mathbf{m}, T_1, T_2$</sub>  (a)  $\mathbf{m} \in K_{\text{ba}}$
- (b)  $\mathfrak{B}_{\mathbf{m}}$  has cardinality  $\leq 2^{\lambda}$
- (c)  $\mathfrak{B}_{\mathbf{m}}$  satisfies the  $\lambda^+$ -c.c. (or just enough to be represented)
- (d)  $\bar{\mathbf{a}}^*$  is a  $(\lambda, \aleph_0, 1, \mathbb{L}(\tau_{T_1}), T_1)$ -moral problem in  $\mathbf{m}$  with no solution in  $\mathbf{m}$
- (e) every  $(\lambda, \aleph_0, 1, \mathbb{L}(\tau_{T_2}), T_2)$ -moral problem has a solution.

2) A sufficient condition for  $\neg(T_1 \leq_{\lambda}^{\text{rp}} T_2)$ :

- (\*\*)  <sub>$T_1, T_2, \lambda$</sub>  (a) there are  $\mathbf{m}_*, \mathbf{a}_*$  satisfying (a), (b), (c), (d) of part (1)
- (b) if  $\langle \mathbf{m}_{\alpha} : \alpha < \delta \rangle$  is  $\leq_{\text{ba}}$ -increasing continuous (so  $D_{\mathbf{m}_{\delta}}$  generated by  $\cup \{D_{\mathbf{m}_{\alpha}} : \alpha < \delta\}$ , as a filter on  $\mathfrak{B}_{\delta}, \delta$  a limit ordinal),  $\bar{\mathbf{a}}^*$  is a  $(\lambda, \aleph_0, 1, \mathbb{L}(\tau_{T_1}), T_1)$ -moral problem in  $\mathbf{m}_0$ , with no solution in  $\mathbf{m}_{\alpha}$  for  $\alpha < \delta$  has no solution in  $\mathbf{m}_{\delta}$
- (c) if  $\mathbf{m} \in K_{\text{ba}}$ ,  $(\mathbf{m}_*, \bar{\mathbf{a}}_*$  are from clause (a))  $\mathbf{m}_* \leq_{\text{ba}} \mathbf{m}$ ,  $\bar{\mathbf{a}}$  is a  $(\lambda, \aleph_0, 1, \mathbb{L}(\tau_{T_2}), T_2)$ -moral problem (and  $\mathfrak{B}_{\mathbf{m}}$  has cardinality  $\leq 2^{\lambda}$ ) and  $\mathbf{n}$  is a simple  $\bar{\mathbf{a}}_*$ -solving extension of  $\mathbf{m}$  then  $\bar{\mathbf{a}}_*$  is not solvable in  $\mathbf{n}$ .

*Proof.* Should be clear, as in the proof of 4.10.  $\square_{5.8}$

**Discussion 5.9.** For  $T_2 = T_{\text{ord}}$  and  $\mu \geq \aleph_0$  by §4 we know there are  $\mathbf{m}_*, \bar{\mathbf{a}}_*$  satisfying clauses (a) + (b) of 5.8(2). So this is a natural starting point for trying to characterize “ $T_2$  is not  $\leq_\lambda^{\text{rp}}$ -maximal”, or at least proved this for some non-trivial cases.

The following shows that dealing with pairs  $(T, \delta)$  and using filters the Theorem of Malliaris-Shelah [MiSh:998] fails.

**Claim 5.10.** 1) *There is a complete first order  $T$  and  $\Delta = \{\varphi(x, y)\}$  where  $\varphi(x, y)$  is an atomic formula in  $\mathbb{L}(\tau_T)$  such that  $(T, \Delta)$  has  $\text{SOP}_2$  and  $\text{NSOP}_3$ .*

2) *Assume  $D$  is a regular filter on  $\lambda$  and  $\mu \leq \lambda$  and  $M$  is a model of  $T$  from part (1).*

*Then the following conditions are equivalent:*

- (a) *the filter  $D$  is a  $(\mu, \aleph_0, 1, \Delta, T)$ -moral filter*
- (b)  *$(\mathbb{Q}, \leq)/D$  has no  $(\kappa, \kappa)$ -pre-cut for any regular  $\kappa = \text{cf}(\kappa) \leq \mu$*
- (c) *in  $(^{\omega > 2}, \triangleleft)^\lambda/D$  every increasing chain of length  $\leq \lambda$  has an upper bound.*

3)  *$(T, \Delta)$  is not  $\triangleleft_{\lambda, \mu}^{\text{rp}}$ -maximal if  $(T, \Delta)$  is the pair constructed in part (1).*

*Proof.* 1) The  $T = \text{Th}(M)$ ,  $\varphi_\ell(x, y) = (xR_\ell y)$  where:

(A)  $\tau = \{P, Q, R\}$  where  $P, Q$  are unary predicates,  $R$  a binary predicate

(B)  $M$  is the following  $\tau$ -model:

- (a) the universe  $^{\omega \geq 2}$
- (b)  $P^M = ^{\omega > 2}, Q^M = ^{\omega 2}$
- (c)  $R^M = \{(a, b) : a \in Q^M, b \in P^M \text{ and } b \triangleleft a\}$ .

2) As in earlier sections.

3) By Theorem 4.10. □<sub>5.10</sub>

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